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Stochastic flows are used to derive martingale representation results and formulae for integration by parts in function space. In turn these, give results on the existence of densities for filtering, smoothing and, prediction problems. Stochastic flows are also used to derive minimum principles in stochastic control, and new equations for the adjoint process. Related results are also obtained for jump processes and the control of Markov chains. Martingale representation results are used to minimize expected risk. Using integration by parts reverse time representations of jump processes are obtained. These results have applications in, for example, smoothing and the Malliavin calculus.

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ROBERT J. ELLIOTT

The Existence of Smooth Densities for the Prediction, Filtering and Smoothing Problems

Brief Outline of Results

A large number of papers have been produced giving results of both practical and theoretical interest. I am grateful for the support of the U.S. Air Force office of Scientific Research and hope it is pleased with what I have done.

In 1987, I realized the central role played by the idea of a stochastic flow; this gave rise to a series of papers. The more theoretical consider martingale representation and integration by parts in function space. This in turn gives rise to more elementary proofs of some results in the Malliavin calculus. These techniques are reported in papers [2], [3] and [8].

I then realized how the proofs could be adapted to show the existence of densities for filtering, smoothing and prediction problems, [1]. In collaboration with John Baras and Michael Kohlman I used the techniques of stochastic flows to obtain results in stochastic control [4], [9], [11]. In particular the martingale representation result was applied and equations for the adjoint process obtained [5] Similar results were obtained for jump processes and reported in [7]. Work with my student Allan Tsoi has included integration by parts formulae and time reversal of jump processes, [16], [20], [25]. With my student Hailiang Yang, results have been obtained on the adjoint process in partially observed control problems [21]. My most interesting recent work contains new equations for the adjoint process. Using martingale representation results it is shown the adjoint process satisfies forward and backward equations, of which the latter is most significant. These results are described in [17], [19], [24]. The adjoint process appears in the minimum principle and so plays a central role in determining any optimal control. Martingale representation results to minimize expected risk are described in [14] and [26].

Full details of results obtained during the contract are presented in the papers.

The Existence of Smooth Densities for the Prediction, Filtering and Smoothing Problems

FINAL REPORT

The following papers have been produced and copies are enclosed:

[1] with M. KOHLMANN, 'The existence of smooth densities for the prediction, filtering and and smoothing problems'. Acta Applic. Math. 14 (1989), 269-286.

[2] with M. KOHLMANN, 'A short proof of a martingale representation result'. Statistics and Probability Letters. 6 (1988), 327-329,

[3] with M. KOHLMANN, 'Integration by parts, homogeneous chaos expansions and smooth densities'. Annals of Probability 17 (1989), 194-207.

[4] with J. BARAS and M. KOHLMANN, 'The partially observed stochastic minimum principle'. S.I.A.M. Jour. Control Opt. 27, (1989), 1279 - 1292.

[5] with M. KOHLMANN, 'The adjoint process in optimal stochastic control'. Lecture Notes in Control and Info. Science, Springer-Verlag. 126 (1989), 115 - 127.

[6] with P.E. KOPP, 'Direct Solutions of Kolmogorov's equations by stochastic flows'. Jour. Math. Anal. App. 142 (1989), 26-34.

[7] with M. KOHLMANN, 'Integration by parts and densities for a jump process'. Stochastics 27 (1989), 83-97.

[8] with M. KOHLMANN, 'Martingale representation and the Malliavin Calculus'. Applied Math. and Optimization. 20(1989), 105-112.

[9] with J. BARAS and M. KOHLMANN, "The conditional adjoint process'. Lecture Notes in Control and Info. Sci. Vol. 111 Springer-Verlag. (1988), 654-662

[10] 'Ordinary differential equations and flows'. Applied Math. Notes. 14 (1989), 1-7.

[11] with M. KOHLMANN, 'The variational principle for optimal control of diffusions with partial information'. Systems and Control Letters. 12 (1989) 63-89.

[12] with M. KOHLMANN and J. MACKI, 'A Proof of the minimum principle using flows'. Annali Polonici. (Accepted)

[13] with M. KOHLMAN, 'Integration by parts and the Malliavin calculus'. Lecture Notes in Control and Info. Sciences. Springer-Verlag. 126 (1989), 128 - 139.

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[15] Filtering for a logistic equation. Mathematical and Computer Modelling. 13 (1990), 1-10.

[16] with A.H. TSOI, Integration by parts for the Poisson processes. Jour. Multivariate Analysis. (Submitted)

[17] The optimal control of diffusions. Applied Math. and Opt. 22 (1990), 229-240.

[18] Martingales associated with finite Markov chains. Vancouver Conference on Markov Processes.

[19] A partially observed control problem for Markov chains. Applied Math. and Opt. (Accepted).

[20] with A.H. TSOI. Integration by parts for the single jump process. Statistics and Probability Letters. (Accepted).

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[22] Filtering and estimation of a Markov chain. 23nd IEEE Asilomar Conference on Signals, Systems & Computers. Asilomar CA. I.E.E.E. Computer Society, Maple Press. 709-713.

[23] with D.SWORDER, An Approximate Finite Dimensional Hybrid Filter. Jour. Optimization Theory and App. (Accepted).

[24] 'The adjoint process for a partially observed Markov chain'. 29th IEEE Control and Decision Conference. (Accepted.)

[25] with A.H. TSOI. Time reversal of non Markov point processes. Annales de l'Institut Henri Poincaré. 26 (1990), 357-373.

[26] with H. FÖLLMER. Orthogonal martingale representation. Festschrift for M. Zakai to be published by Academic Press in 1991.

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The existence of smoooth densities for the prediction, filtering and smoothing problem

by

Dr. Robert J. Elliott

Acta Applicandae Mathematicae 14 (1989), 269–286.

The Existence of Smooth Densities for the Prediction Filtering and Smoothing Problems

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Abstract. Using a simple martingale representation result, a partial integration-by-parts formula is obtained. Quoting the results of Bismut and Michel, it then follows that under Hörmander's conditions on the coefficient vector fields, the filtering, smoothing and prediction problems have C^∞ density solutions. The paper does not require the development of any analysis over Wiener space.

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Key words. Filtering, prediction, smoothing, Malliavin calculus, stochastic differential equations.

1. Introduction

Following Malliavin's remarkable work [8], there have been other treatments of the Malliavin calculus, including those of Bismut [1], Stroock [11] and Norris [10]. A particularly readable account can be found in the paper of Zakai [13]. In [2], Bismut and Michel developed a conditional version of the Malliavin calculus to show the existence of a conditional density in filtering and smoothing problems. Other important applications of the Malliavin calculus to filtering problems include the work of Cattiaux [4], Kusuoka and Stroock [11] and Michel [9]. Using a simple and natural expression for the integrand in a stochastic integral, the authors [5] have been able to give an elementary proof of the existence of a density for a diffusion under Hörmander's conditions for the coefficient vector fields. The homogeneous chaos expansion of the random variable is also obtained in [5]. The objective of this paper is to present a conditional version of the results of [5] and, following the exposition of Zakai, simplify some of the results of

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Bismut and Michel. In particular, a conditional integration-by-parts formula is obtained without using any functional analysis over Wiener space. However, the delicate technical sufficient conditions for the integrability of the inverse of the Malliavin matrix are not discussed. For these we refer to Bismut and Michel [2].

In this paper, the following system of stochastic differential equations is considered:

$$dx = X_0(x, y)\,dt + X_i(x, y)\,dw^i + \bar{X}_j(x, y)\,dB^j + \bar{X}_j(x, y)h^j(x, y)\,dt,$$

$$dy = Y_0(y)\,dt + Y_j(y)\,dB^j + Y_j(y)h^j(x, y)\,dt.$$

Here $w = (w^1, \ldots, w^n)$ and $B = (B^1, \ldots, B^n)$ are independent Brownian motions. The process x represents the unobserved signal process, while y represents the observation process. If $\{Y_t\}$ is the right continuous, complete filtration generated by $\{y_t\}$, then the filtering problem discusses $E[x_t \mid Y_t]$, the prediction problem discusses $E[x_t \mid Y_s]$ when $s \leq t$, and the smoothing problem discusses $E[x_t \mid Y_s]$ when $s \geq t$.

Using the simple martingale representation result of [5], a conditional version of the Malliavin calculus is developed in Section 4. Suppose $T \geq t$ and let c be any smooth function on R^d with bounded derivatives of all orders. In Section 5, we show that if the inverse of the conditional Malliavin matrix $M_{0,T}$ belongs to $L^p(\Omega)$ for all p, $1 \leq p < \infty$, then

$$\left| E\left[\frac{\partial^\alpha c}{\partial x^\alpha}(x_t) \mid Y_T\right]\right| \leq K(y) \sup_{x \in R^d} |c(x)|$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $K(y)$ is a Y_T-measurable random variable which is finite a.s.

This inequality, using simple Fourier analysis, implies that the random variable x_t has almost surely a conditional density given Y_T, which is infinitely differentiable. Using Jensen's inequality we can immediately deduce

$$\left| E\left[\frac{\partial^\alpha c}{\partial x^\alpha}(x_t) \mid Y_s\right]\right| \leq K'(y) \sup_{x \in R^d} |c(x)|,$$

where $s \geq t$ or $s \leq t$. Therefore, the smoothing, filtering and prediction problems for x_t, given Y_s, have, almost surely, smooth conditional density solutions.

2. Stochastic Flows

We recall in this section the properties of stochastic flows, and in particular those relating to 'lower triangular' systems obtained by Norris [10]. See also Stroock [11]. Let $w_t = (w_t^1, \ldots, w_t^n)$, $t \geq 0$, be an n-dimensional Brownian motion on (Ω, F, P). Write $\{F_t\}$ for the right continuous, complete filtration generated by w. Suppose $X_0, X_1, \ldots, X_m$ are smooth vector fields on $[0, \infty] \times R^d$, all of whose derivatives are bounded. Then from Bismut [1], or Carverhill and Elworthy [3], we quote the following result:

THEOREM 2.1. *There is a map* $\xi:\Omega\times[0,\infty)\times[0,\infty)\times R^d\to R^d$ *such that (i) for* $0\leq s\leq t$ *and* $x\in R^d$ $\xi_{s,t}(x)$ *is the essentially unique solution of the stochastic differential equation*

$$d\xi_{s,t}(x)=X_0(t,\xi_{s,t}(x))\,dt+X_i(t,\xi_{s,t}(x))\,dw_t^i, \tag{2.1}$$

with $\xi_{s,s}(x)=x$. *(Note the Einstein summation convention is used.)*

(ii) For each ω, s, t *the map* $\xi_{s,t}(\cdot)$ *is* C^∞ *on* R^d *with a first derivative, the Jacobian,* $\partial\xi_{s,t}/\partial x=D_{s,t}$, *which satisfies*

$$dD_{s,t}=\frac{\partial X_0}{\partial\xi}(t,\xi_{s,t}(x))D_{s,t}\,dt+\frac{\partial X_i}{\partial\xi}(t,\xi_{s,t}(x))D_{s,t}\,dw_t^i \tag{2.2}$$

with initial condition $D_{s,s}=I$, *the* $d\times d$ *identity matrix.*

REMARKS 2.2. This result is proved, as quoted, for possibly time inhomogeneous coefficient vector fields, X_i, though we shall not use this generality. Note that (2.2) is obtained formally by differentiating (2.1). In fact, equations for higher derivatives $\partial^n\xi/\partial x^n$ are obtained by further differentiation. However, if we consider the enlarged system given by (2.1) and (2.2), the coefficients are not bounded, because of the linear appearance of $D_{s,t}$ on the right of (2.2). However, Norris [10] has extended the results of Theorem 2.1 to such systems with time homogeneous coefficients. To state Norris's results, we first define a class of 'lower triangular' coefficients.

DEFINITION 2.3. For positive integers α, d, $d_1,\ldots,d_k$ write $S_\alpha(d_1,\ldots,d_k)$ for the set of $X\in C^\infty(R^d,R^d)$ *of the form*

$$X(x)=\begin{pmatrix}X^{(1)}(x^1)\\ X^{(2)}(x^1,x^2)\\ \vdots\\ X^{(k)}(x^1,x^2,\ldots,x^k)\end{pmatrix}\text{ for } x=\begin{pmatrix}x^1\\ x^2\\ \vdots\\ x^k\end{pmatrix} \tag{2.3}$$

where R^d is identified with $R^{d_1}\times\cdots\times R^{d_k}$, $x^j\in R^{d_j}$ and the X satisfy

$$\|X\|_{S(\alpha,N)}=\sup_{x\in R^d}\left(\sup_{0\leq n\leq N}\frac{|D^nX(x)|}{(1+|x|^n)}\vee\sup_{1\leq j\leq k}|D_jX^{(j)}(x)|\right)$$
$$<\infty\text{ for all positive integers } N. \tag{2.4}$$

Write $S(d_1,\ldots,d_k)=\bigcup_\alpha S_\alpha(d_1,\ldots,d_k)$.

REMARKS 2.4. Note Equations (2.1) and (2.2) can be considered as a single system whose coefficients are not bounded, but are in $S(d,d^2)$. The final supremum on the right of (2.4) implies the first derivatives of $X^{(1)}$ are bounded, as are the first derivatives D_j in the 'new' variable x^j of $X^{(j)}(x^1,\ldots,x^j)$. This means $X^{(j)}$ is allowed linear growth in x^j, a situation illustrated in (2.2). We quote from Norris the following result.

THEOREM 2.5. *Let* $X_0, X_1, \ldots, X_m \in S_\alpha(d_1, \ldots, d_k)$. *Then there is a map* $\phi: \Omega \times [0, \infty) \times [0, \infty) \times R^d \to R^d$ *such that*

(*i*) *for* $0 \leq s \leq t$ *and* $x \in R^d$ $\phi(\omega, s, t, x)$ *is the essentially unique solution of the stochastic differential equation*

$$\mathrm{d}x_t = X_0(x_t)\,\mathrm{d}t + X_i(x_t)\,\mathrm{d}w_t^i \tag{2.5}$$

with $x_s = x$.

(*ii*) *for each* ω, s, t *the map* $\phi(\omega, s, t, x)$ *is* C^∞ *in* x *with derivatives of all orders satisfying stochastic differential equations obtained from* (2.5) *by formal differentiation.*

(*iii*)
$$\sup_{|x|\leq R} E[\sup_{s\leq u\leq t} |D^N\phi(\omega, s, u, x)|^p] \leq C(p, s, t, R, N, d_1, \ldots, d_k, \alpha, \|X_0\|_{S(\alpha,N)}, \ldots, \|X_n\|_{S(\alpha,N)}). \tag{2.6}$$

REMARKS 2.6. Norris proves Theorem 2.5 by induction on i. Write (2.5) as a system of stochastic differential equations for $j = 1, \ldots, k$

$$\mathrm{d}x_t^j = X_0^{(j)}(x_t^1, \ldots, x_t^j)\,\mathrm{d}t + X_i^{(j)}(x_t^1, \ldots, x_t^j)\,\mathrm{d}w_t^i,$$
$$x_s^j = x^j \in R^{d_j}. \tag{2.7}$$

Suppose the result is true for $i = 1, \ldots, j-1$ and write $\tilde{X}_i^{(j)}(\omega, s, t, x^j) = X_i^{(j)}(x_t^1(\omega), \ldots, x_t^{j-1}(\omega), x^j)$. Then (2.7) can be written in the form

$$\mathrm{d}x_t^j = \tilde{X}_0(s, t, x_t^j)\,\mathrm{d}t + \tilde{X}_i(s, t, x_t^j)\,\mathrm{d}w_t^i$$

and Theorem 2.1 applied. A difficult step is establishing the result for $j = 1$. However, this follows by a stopping time argument, which is essentially the method by Bismut [1]. Returning to the, possibly time inhomogeneous, situation of Theorem 2.1, consider the process V defined by

$$\mathrm{d}V_{s,t} = -V_{s,t}\left(\frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x)) - \sum_{i=1}^{n}\left(\frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))\right)^2\mathrm{d}t - \right.$$
$$\left. - V_{s,t}\frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))\,\mathrm{d}w_t^i, \right. \tag{2.8}$$

with $V_{s,s} = I$. Then by applying the Ito rule, we see $d(D_{s,t}V_{s,t}) = 0$, while $D_{s,s}V_{s,s} = I$, the $d \times d$ identity matrix. Therefore, $V_{s,t} = D_{s,t}^{-1}$. By applying Theorem 2.5(iii) to the system given by Equations (2.1), (2.2) and (2.8), we have

$$|D_{s,t}^*| = \sup_{s\leq u\leq t} |D_{s,u}| \quad \text{and} \quad |V_{s,t}^*| = \sup_{s\leq u\leq t} |V_{s,u}|$$

are in $L^p(\Omega)$ for all $p < \infty$. Finally, for $0 \leq s \leq t$, recall, by the uniqueness of the solution of (2.1):

$$\xi_{0,t}(x_0) = \xi_{s,t}(\xi_{0,s}(x_0)) = \xi_{s,t}(x), \quad \text{if } x = \xi_{0,s}(x_0). \tag{2.9}$$

Differentiating (2.9)

$$D_{0,t} = D_{s,t} D_{0,s} \tag{2.10}$$

and

$$V_{0,t} = V_{0,s} V_{s,t}. \tag{2.11}$$

3. Martingale Representation

Consider a stochastic differential system with coefficients in some set S, as discussed in Theorem 2.5, and let $\xi_{0,t}(x, 0)$ be its stochastic flow solution. For some $T > 0$ consider a real-valued differential function c for which the random variable $c(\xi_{0,T}(x_0))$ and the components of the gradient $c_\xi(\xi_{0,T}(x_0))$ are integrable. Let M_t be the right continuous version of the martingale

$$E[c(\xi_{0,T}(x_0)) \mid F_t].$$

There exist several proofs of martingale representation results; see, for example, Bismut [1] and the references given there. However, the following proof in the Markov case, see [5], is particularly straightforward.

THEOREM 3.1. *For* $0 \le t \le T$, $M_T = E[c(\xi_{0,T}(x_0))] + \int_0^t \gamma_i(s)\,\mathrm{d}w_s^i$ *where*

$$\gamma_i(s) = E[c_\xi(\xi_{0,T}(x_0))\, D_{0,T} \mid F_s]\, D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)).$$

Proof. It is well known that M_t has a representation

$$M_t = M_0 + \int_0^t \gamma_i(s)\,\mathrm{d}w_s^i \tag{3.1}$$

for some predictable integrands γ_i. Because the process $\xi_{0,T}(x_0)$ is Markov

$$\begin{aligned} M_t &= E[c(\xi_{0,T}(x_0)) \mid F_t] \\ &= E[c(\xi_{t,T}(x)) \mid F_t] \\ &= E_t[c(\xi_{t,T}(x))] \\ &= V(t, x), \text{ say, where } x = \xi_{0,t}(x_0). \end{aligned} \tag{3.2}$$

By Theorem 2.5 and the chain rule $c(\xi_{t,T}(x))$ is differentiable, in fact smooth, in x. The differentiability of $E_{t,x}[c(\xi_{t,T}(x))]$ in t can be established by writing the backward equation for $\xi_{t,T}(x)$, as in Kunita [6]. Consequently, applying the Ito role to $V(t, x)$, with $x = \xi_{0,T}(x_0)$ we have

$$\begin{aligned} V(t, \xi_{0,t}(x_0)) = V(0, x_0) &+ \int_0^t \left(\frac{\partial V}{\partial s} + LV\right) \mathrm{d}s + \\ &+ \int_0^t \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0)) X_i(s, \xi_{0,s}(x_0))\,\mathrm{d}w_s^i \end{aligned} \tag{3.3}$$

where

$$L = \sum_{i=1}^{d} X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\sum_{k=1}^{m} X_k^i X_k^j \right) \frac{\partial^2}{\partial x_i \, \partial x_j}.$$

By the uniqueness of the decomposition of special semimartingales, comparing (3.1) and (3.3), we must have (as is well known)

$$\frac{\partial V}{\partial s} + LV = 0$$

and

$$\gamma_i(s) = \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0)) X_i(s, \xi_{0,s}(x_0)).$$

From (3.2)

$$\frac{\partial V}{\partial x} = E[c_\xi(\xi_{s,T}(x))\, D_{s,T} \mid F_s]$$

so by (2.10)

$$\gamma_i(s) = E[c_\xi(\xi_{0,T}(x_0))\, D_{0,T} \mid F_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)).$$

COROLLARY 3.2. *The result extends immediately to vector (or matrix) functions* c.

COROLLARY 3.3. *Note, in particular,*

$$c(\xi_{0,T}(x_0)) = E[c(\xi_{0,T}(x_0))] + \int_0^T E[c_\xi(\xi_{0,T}(x_0)) D_{0,T} \mid F_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0))\, dw_s^i. \qquad (3.4)$$

LEMMA 3.4. F_t *is generated by the set of stochastic integrals of the form* $\int_0^t \gamma_i(s, w_s)\, d\omega_s^i$, *where the integrands* γ_i *are smooth functions of* s *and* w_s *at time* s, *with bounded derivatives of all orders.*

Proof. $\sigma\{w_t\}$ is generated by $g(w_t)$ for $g \in C_b^\infty(R^d)$. If we apply Theorem 3.1 to the process w_t, so $w_t = x + (w_t - w_s)$ where $x = w_s$, the Jacobian is the identity I and

$$E[g_w(w_t) \mid F_s] = E_{s,x}[g_w(w_t)] = \gamma(w_s),$$

where

$$\gamma(w_s) = (\gamma_1(w_s), \ldots, \gamma_m(w_s)) = E_{s,w_s}[(g_{w^1}(w_t), \ldots, g_{w^m}(w_t))].$$

Therefore,

$$g(w_t) = E[g(w_t)] + \int_0^t \gamma_i(w_s)\, dw_s^i.$$

where the $\gamma_i \in C_b^\infty(R^d)$. Consequently, $\sigma(w_t)$ is generated by stochastic integrals of this form. Allowing the integrands to depend on s we that F_t, which is generated by w_s, for $s \leq t$, is generated by stochastic integrals of the form $\int_0^t \gamma_i(s, w_s)\, dw_s^i$, where $\gamma_i \in C_b^\infty([0, \infty) \times R^n)$.

REMARKS 3.5. So far we have considered an n-dimensional Brownian motion $w = (w^1, \ldots, w^m)$ and a state vector $x \in R^d$. Consider now a larger system: suppose $B = (B^1, \ldots, B^n)$ is an n-dimensional Brownian motion, defined on a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$, which is independent of w. Write $\{\bar{F}_t\}$ for the right continuous, complete filtration generated by B, and $\{G_t\}$ for the right continuous, complete filtration of $\Omega \times \bar{\Omega}$ generated by $F_t \times \bar{F}_t$. Consider a second state vector $y \in R^p$ and a stochastic differential system defined on $(\Omega \times \bar{\Omega}, F \times \bar{F}, P \times \bar{P})$ by the equations

$$\begin{aligned} dx_t &= X_0(x_t, y_t)\, dt + X_i(x_t, y_t)\, dw_t^i + \tilde{X}_j(x_t, y_t)\, dB_t^j, \\ dy_t &= Y_0(y_t)\, dt + Y_j(y_t)\, dB_t^j, \end{aligned} \tag{3.5}$$

with $(x(0), y(0)) = (x_0, y_0) \in R^d \times R^p$. We shall suppose the coefficient vector fields $X_0, \ldots, X_m, Y_0, \ldots, Y_n$ are such that the coefficients of (3.5) belong to the space S, so that Theorem 2.5 can be applied. Note that in (3.5) the process y is not influenced by the process x.

NOTATION 3.6. Suppose $(x, y) \in R^d \times R^p$ is the state of the system (3.5) at time s. We shall denote the solution flow of (3.5) for $t \geq s$ by the map

$$(x, y) \to (x_{s,t}(x, y), y_{s,t}(y)).$$

The Jacobian of this map looks like

$$\frac{\partial(x_{s,t}(x, y), y_{s,t}(y))}{\partial(x, y)} = \begin{pmatrix} \dfrac{\partial(x_{s,t}(x, y))}{\partial x} & \dfrac{\partial(x_{s,t}(x, y))}{\partial y} \\ 0 & \dfrac{\partial(y_{s,t}(y))}{\partial y} \end{pmatrix}. \tag{3.6}$$

Write $D_{s,t}(x, y)$ for the 'partial' Jacobian $\partial(x_{s,t}(x, y))/\partial x$. The existence of the large Jacobian, and, therefore, of its components, including $D_{s,t}$, is given by Theorem 2.5.

As in [2], we now introduce a new measure on $(\Omega \times \bar{\Omega}, F \times \bar{F})$ by a Girsanov change of density.

NOTATION 3.7. Suppose $h(x, y) = (h^1(x, y), \ldots, h^n(x, y))$ is a smooth function in $C^\infty(R^{d+p}, R^n)$ with bounded derivatives of all orders. Define the real valued process L on $\Omega \times \bar{\Omega} \times [0, \infty)^2 \times R^d \times R^p$ by

$$\begin{aligned} &L_{s,t}(x, y) \\ &\quad = \exp\left\{\int_s^t h^j(x_{s,u}(x, y), y_{s,u}(y))\, dB_u^j - \frac{1}{2}\sum_{j=1}^n \int_s^t h^j(x_{s,u}(x, y), y_{s,u}(y))^2\, du\right\}. \end{aligned}$$

Then

$$\mathrm{d}L_{s,t}(x,y) = L_{s,t}(x,y)h^j(x_{s,t}(x,y), y_{s,t}(x,y))\,\mathrm{d}B_t^j, \tag{3.7}$$

with $L_{s,s}(x,y) = 1$, so L is a $\{G_t\}$ martingale. Furthermore, $L^*_{0,t} = \sup_{u \le t} L_{0,u}$ is in every space $L^p(\Omega)$, $1 \le p < \infty$. Because h is bounded, we also have that $(L_{0,t}^{-1})^* = \sup_{u \le t}(L_{0,u}^{-1})$ is in every $L^p(\Omega)$, $1 \le p < \infty$. We could consider the flow given by the combined system (3.5) and (3.7). However, for the moment note that for $0 \le s \le t$

$$L_{0,t}(x_0, y_0) = L_{0,s}(x_0, y_0)L_{s,t}(x,y), \tag{3.8}$$

so writing $L = L_{0,s}(x_0, y_0)$ we have

$$\frac{\partial L_{0,t}}{\partial L} = L_{s,t}(x,y)$$

and

$$\frac{\partial L_{0,t}}{\partial x_0} = \frac{\partial L_{0,s}}{\partial x_0} L_{s,t}(x,y) + L_{0,s}\frac{\partial L_{s,t}}{\partial x} D_{0,s} \tag{3.9}$$

with a similar equation for $\partial L_{0,t}/\partial y_0$.

DEFINITION 3.8. Define a measure P_h on $(\Omega \times \bar{\Omega}, F \times \bar{F})$ such that its restriction to G_t is given by

$$\mathrm{d}P_h(\omega, \bar{\omega}) = L_{0,t}(x_0, y_0)\,\mathrm{d}P(\omega) \times \mathrm{d}\bar{P}(\bar{\omega}).$$

Then Girsanov's theorem states:

THEOREM 3.9. *Under P_h the process B' is an n-dimensional Brownian motion independent of w, where*

$$B'_t = B_t - \int_0^t h(x_{0,s}, y_{0,s})\,\mathrm{d}s.$$

Therefore, under the measure P_h, the process $(x_{s,t}, y_{s,t})$ is the solution of the stochastic differential equation

$$\begin{aligned}\mathrm{d}x_{s,t} &= X_0(x_{s,t}, y_{s,t})\,\mathrm{d}t + X_i(x_{s,t}, y_{s,t})\,\mathrm{d}w_t^i + \tilde{X}_j(x_{s,t}, y_{s,t})\,\mathrm{d}B_t'^j + \\ &\quad + \tilde{X}_j(x_{s,t}, y_{s,t})h^j(x_{s,t}, y_{s,t})\,\mathrm{d}t, \\ dy_{s,t} &= Y_0(y_{s,t})\,\mathrm{d}t + Y_j(y_{s,t})\,\mathrm{d}B_t'^j + Y_j(y_{s,t})h^j(x_{s,t}, y_{s,t})\,\mathrm{d}t,\end{aligned} \tag{3.10}$$

with $(x_{s,s}, y_{s,s}) = (x, y) \in R^d \times R^p$.

REMARKS 3.10. The system (3.10) provides a natural setting in which to discuss filtering, smoothing or prediction problems. The process x_t represents a signal which is not observed directly. Instead, one observes the process y_t which is influenced by x_t through the process $h(x_t, y_t)$. Write $\{Y_t\}$ for the right continuous,

complete filtration generated by y, and E_h for expectation under P_h. The filtering problem discusses $E_h[x_t \mid Y_t]$, the smoothing problem discusses $E_h[x_t \mid Y_T]$, where $t \leq T$, and the prediction problem discusses $E_h[x_t \mid Y_T]$, where $t \geq T$.

In this paper, using the techniques of the Malliavin calculus, we given sufficient conditions in the filtering, smoothing and prediction cases, that the conditional distribution of x_t has a smooth density.

4. Integration by Parts

Suppose $0 < t \leq T$ and let $U_{0,T}(\bar{\omega})$ be an $\bar{F}_T$ measurable random variable of the form discussed in Lemma 3.4, that is

$$U_{0,T}(\bar{\omega}) = \int_0^T \gamma_j(s, B_s)\, \mathrm{d}B_s^j, \tag{4.1}$$

where $\gamma_j \in C_b^\infty([0, \infty) \times R^n)$ for $1 \leq j \leq n$. Consider the system given by (3.5), (3.7) and (4.1) on $(\Omega \times \bar{\Omega}, F \times \bar{F}, P \times \bar{P})$:

$$\begin{aligned}
\mathrm{d}x_{s,t} &= X_0(x_{s,t}, y_{s,t})\, \mathrm{d}t + X_i(x_{s,t}, y_{s,t})\, \mathrm{d}w_t^i + \tilde{X}_j(s_{s,t}, y_{s,t})\, \mathrm{d}B_t^j, \\
\mathrm{d}y_{s,t} &= Y_0(y_{s,t})\, \mathrm{d}t + Y_j(y_{s,t})\, \mathrm{d}B_t^j, \\
\mathrm{d}L_{s,t} &= L_{s,t} h^j(x_{s,t}, y_{s,t})\, \mathrm{d}B_t^j, \\
\mathrm{d}U_{s,t} &= \gamma_j(t, B_t)\, \mathrm{d}B_t^j.
\end{aligned} \tag{4.2}$$

Then Theorem 2.5, with $(x_{s,s}, y_{s,s}, L_{s,s}, U_{s,s}) = (x, y, 1, 0)$, can be applied to (4.2) and we can consider the associated stochastic flow. Note $U_{s,t}$ does not involve x, y, or L, and if $U_{0,s} = U$ then

$$U_{0,t} = U + \int_s^t \gamma_j(s, B_s)\, \mathrm{d}B_s^j. \tag{4.3}$$

Also, if $L = L_{0,s}$, from (3.8)

$$L_{0,t} = L L_{s,t}(s, y). \tag{4.4}$$

THEOREM 4.1. *Suppose $0 < t \leq T$ and let c be a C^∞ function on R^d with bounded derivatives of all orders. Then for any square integrable predictable process $u(s) = (u_1(s), \ldots, u_m(s))$*

$$\begin{aligned}
&E\left[U_{0,T} L_{0,T} c(x_{0,t}(x_0, y_0)) \int_0^T u_i(s)\, \mathrm{d}w_s^i \right] \\
&\quad = \sum_{i=1}^m E\left[U_{0,T} L_{0,T} c_x(x_{0,t}(x_0, y_0)) D_{0,t} \int_0^t D_{0,s}^{-1} X_i(s) u_i(s)\, \mathrm{d}s \right] + \\
&\quad + \sum_{i=1}^m E\left[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t}(x_0, y_0)) \frac{\partial L_{0,T}}{\partial x_0} \int_0^T D_{0,s}^{-1} X_i(s) u_i(s)\, \mathrm{d}s \right]
\end{aligned}$$

$$-\sum_{i=1}^{m} E\left[U_{0,T}L_{0,T}c(x_{0,t}(x_0, y_0))\int_0^T L_{0,s}^{-1}\frac{\partial L_{0,s}}{\partial x_0}D_{0,s}^{-1}X_i(s)u_i(s)\,\mathrm{d}s\right]. \quad (4.5)$$

Proof. First recall the derivation of Theorem 3.1 and write for $0 \leq s \leq t \leq T$

$$\begin{aligned} V(s, x, y, L, U) &= E[U_{0,T}L_{0,T}(x_0, y_0)c(x_{0,t}(x_0, y_0)) \mid G_s] \\ &= E[(U + U_{s,T})LL_{s,T}(x, y)c(x_{s,t}(x, y)) \mid G_s] \qquad (4.6) \\ &= E_{s,x,y,L,U}[(U + U_{s,T})LL_{s,T}(x, y)c(x_{s,t}(x, y))]. \end{aligned}$$

The martingale representation result is obtained by writing down the Ito formula for V, and the derivatives of V are found by differentiating the conditional expectation (4.5) in x, y, L and U. Note that for $s > t$ the derivative of $c(x_{s,t}(x, y))$ in x is zero. We, therefore, have

$$\begin{aligned} &U_{0,T}L_{0,T}(x_0, y_0)c(x_{0,t}(x_0, y_0)) \\ &= E[U_{0,T}L_{0,T}(x_0, y_0)c(x_{0,t}(x_0, y_0))] + \\ &+ \int_0^t E[U_{0,T}L_{0,T}c_x(x_{0,t}(x_0, y_0))D_{0,t} \mid G_s]D_{0,s}^{-1}(X_i\,\mathrm{d}w_s^i + \bar{X}_j\,\mathrm{d}B_s^j) + \\ &+ \int_0^t E\left[U_{0,T}L_{0,T}c_x(x_{0,t}(x_0, y_0))\frac{\partial(x_{s,t}(x, y))}{\partial y} \,\Big|\, G_s\right]Y_j\,\mathrm{d}B_s^j + \\ &+ \int_0^T E[U_{0,T}L_{s,T}c(x_{0,t}(x_0, y_0) \mid G_s]h^j\,\mathrm{d}B_s^j + \\ &+ \int_0^T E\left[U_{0,T}L\frac{\partial L_{s,T}}{\partial x}(x, y)c(x_{0,t}(x_0, y_0)) \,\Big|\, G_s\right](X_i\,\mathrm{d}w_s^i + \bar{X}_j\,\mathrm{d}B_s^j) + \\ &+ \int_0^T E\left[U_{0,T}L\frac{\partial L_{s,T}}{\partial y}(x, y)c(x_{0,t}(x_0, y_0)) \,\Big|\, G_s\right]Y_j\,\mathrm{d}B_s^j + \\ &+ \int_0^T E[U_{s,T}L_{0,T}c(x_{0,t}(x_0, y_0)) \mid G_s]\gamma_j\,\mathrm{d}B_s^j. \qquad (4.7) \end{aligned}$$

Taking the product of (4.7) with $\int_0^T u_i(s)\,\mathrm{d}w_s^i$, because w and B are independent under $P \times \bar{P}$, we have

$$\begin{aligned} &E\left[U_{0,T}L_{0,T}c(x_{0,t})\int_0^T u_i(s)\,\mathrm{d}w_s^i\right] \\ &= \sum_{i=1}^{m} E\left[U_{0,T}L_{0,T}c_x(x_{0,t})D_{0,t}\int_0^t D_{0,s}^{-1}X_i(s)u_i(s)\,\mathrm{d}s\right] + \\ &+ \sum_{i=1}^{m} E\left[U_{0,T}c(x_{0,t})\int_0^T L\frac{\partial L_{s,T}}{\partial x}(x, \ )X_i(s)\,\mathrm{d}s\right] \qquad (4.8) \end{aligned}$$

From (3.9),

$$L\frac{\partial L_{s,T}}{\partial x} = \frac{\partial L_{0,T}}{\partial x_0} D_{0,s}^{-1} - \frac{\partial L_{0,s}}{\partial x_0} L_{s,T}(x, y) D_{0,s}^{-1}.$$

Substituting in (4.8) the result follows.

NOTATION 4.2. Write * for the transpose. Furthermore, write

$$R_{0,T} = \int_0^T (D_{0,s}^{-1} X_i(s))^* \, \mathrm{d}w_s^i,$$

$$\Lambda_{0,T} = \sum_{i=1}^m \int_0^T L_{0,s}^{-1} \frac{\partial L_{0,s}}{\partial x_0} D_{0,s}^{-1} X_i(s) X_i(s)^* (D_{0,s}^{-1})^* \, \mathrm{d}s$$

and recall the Malliavin matrix, [1], [5], (which here is a 'partial' Malliavin matrix in the X_i vector fields):

$$M_{0,T} = \sum_{i=1}^m \int_0^T D_{0,s}^{-1} X_i(s) X_i(s)^* (D_{0,s}^{-1})^* \, \mathrm{d}s.$$

COROLLARY 4.3. *We then have the special case of Theorem 4.1 obtained by taking* $u_i(s) = (D_{0,s}^{-1} X_i(s))^*$:

$$\begin{aligned} &E[U_{0,T} L_{0,T} c(x_{0,t}) R_{0,T}] \\ &\quad = E[U_{0,T} L_{0,T} c_x(x_{0,t}) D_{0,t} M_{0,t}] + \\ &\qquad + E\left[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T}\right] - \\ &\qquad - E[U_{0,T} L_{0,T} c(x_{0,t}) \Lambda_{0,T}] \end{aligned} \tag{4.9}$$

COROLLARY 4.4. *Equation* (4.9) *is still true for vector, (or matrix), functions c.*

REMARKS 4.5 The gradient c_x of c occurs in only one term, so (4.9) is an 'integration by parts' formula. Suppose g is a second smooth function with bounded derivatives of all orders. Applying (4.9) to the product $c(x_{0,t})g(x_{0,t})$ we have

$$\begin{aligned} &E[U_{0,T} L_{0,T} c(x_{0,t}) g(x_{0,t}) R_{0,T}] \\ &\quad = E[U_{0,T} L_{0,T} (c_x(x_{0,t}) g(x_{0,t}) + c(x_{0,t}) g_x(x_{0,t})) D_{0,t} M_{0,t}] + \\ &\qquad + E\left[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t}) g(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T}\right] \\ &\qquad - E[U_{0,T} L_{0,T} c(x_{0,t}) g(x_{0,t}) \Lambda_{0,T}]. \end{aligned} \tag{4.10}$$

From Lemma 3.4 the random variables $U_{0,T}$ generate $\tilde{F}_T$ so (4.10) can be written

$$E[L_{0,T}c(x_{0,t})g(x_{0,t})R_{0,T} \mid \tilde{F}_T]$$
$$= E[L_{0,T}(c_x(x_{0,t})g(x_{0,t}) + c(x_{0,t})g_x(x_{0,t}))D_{0,t}M_{0,t} \mid \tilde{F}_T] +$$
$$+ E\left[L_{0,T}L_{0,T}^{-1}c(x_{0,t})g(x_{0,t})\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid \tilde{F}_T\right]$$
$$- E[L_{0,T}c(x_{0,t})g(x_{0,t})\Lambda_{0,T} \mid \tilde{F}_T].$$

Under $P \times \tilde{P}$, $Y_T \subset \tilde{F}_T$ so

$$E[L_{0,T}c(x_{0,t})g(x_{0,t})R_{0,T} \mid Y_T]$$
$$= E[L_{0,T}(c_x(x_{0,t})g(x_{0,t}) + c(x_{0,t})g_x(x_{0,t}))D_{0,t}M_{0,t} \mid Y_T] +$$
$$+ E\left[L_{0,T}L_{0,T}^{-1}c(x_{0,t})g(x_{0,t})\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid Y_T\right]$$
$$- E[L_{0,T}c(x_{0,t})g(x_{0,t})\Lambda_{0,T} \mid Y_T].$$

Now

$$E_h[c(x_{0,t})g(x_{0,t})R_{0,T} \mid Y_T] = E[L_{0,T}c(x_{0,t})g(x_{0,t})R_{0,T} \mid Y_T](E[L_{0,T} \mid Y_T])^{-1}.$$

Furthermore, $L_{0,T} > 0$ a.s.; therefore

$$E[L_{0,T} \mid Y_t]^{-1} < \infty \text{ a.s.}$$

Consequently, dividing by $E[L_{0,T} \mid Y_T]$ we have

$$E_h[c(x_{0,t})g(x_{0,t})R_{0,T} \mid Y_T]$$
$$= E_h[(c_x(x_{0,t})g(x_{0,t}) + c(x_{0,t})g_x(x_{0,t}))D_{0,t}M_{0,t} \mid Y_T] +$$
$$+ E\left[L_{0,T}^{-1}c(x_{0,t})g(x_{0,t})\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid Y_T\right]$$
$$- E_h[c(x_{0,t})g(x_{0,t})\Lambda_{0,T} \mid Y_T], \tag{4.11}$$

where both sides are finite a.s.

With this in mind, to obtain a bound for the conditional expectation $E_h[c_x(x_{0,t}) \mid Y_T]$ we would like to take $g = M_{0,t}^{-1}D_{0,t}^{-1}$ in (4.11). However, $D_{0,t}$ and $M_{0,t}$ involve the past of the process $\xi_{0,t}$, $D_{0,t}$ and $M_{0,t}$. This difficulty can be circumvented by considering an enlarged system. A second difficulty is that the function $g(M_{0,t}, D_{0,t}) = M_{0,t}^{-1}D_{0,t}^{-1}$ does not have bounded derivatives. However, $D^{-1} = V$ is given by (2.8). Considering $g_\epsilon(M, V) = (M + \epsilon)^{-1}V$ for $\epsilon > 0$ and letting $\epsilon \to 0$ we see Equation (4.11) holds for such a g.

NOTATION 4.6. Let $\phi^{(0)}(\omega, \tilde{\omega}, s, t, x, y, L, U)$ denote the flow associated with the system (4.2). Write $D_{s,t}^{(0)}$ for the Jacobian associated with this flow. Note that among the components of $D_{s,t}^{(0)}$ are the 'partial' Jacobian

$$\frac{\partial x_{s,t}(x,y)}{\partial x} = D_{s,t}$$

and the gradient $\partial L_{s,t}(x,y)/\partial x$. Write

$$R^{(0)}_{s,t} = R_{s,t} = \int_s^t (D^{-1}_{s,u} X_i(u))^* \, \mathrm{d}w^i_u,$$

$$\Lambda^{(0)}_{s,t} = \Lambda_{s,t} = \sum_{i=1}^m \int_s^t L^{-1}_{s,u} \frac{\partial L_{s,u}}{\partial x_s} D^{-1}_{s,u} X_i(u) X_i(u)^* (D^{-1}_{s,u})^* \, \mathrm{d}u,$$

$$M^{(0)}_{s,t} = M_{s,t} = \sum_{i=1}^m \int_s^t D^{-1}_{s,u} X_i(u) X_i(u)^* (D^{-1}_{s,u})^* \, \mathrm{d}u.$$

Then the system

$$\phi^{(1)} = (\phi^{(0)}, D^{(0)}, R^{(0)}, M^{(0)}, \Lambda^{(0)})$$

is Markov with coefficients in

$$S(d+p+2,\ d+p+2+(d+p+2)^2,$$
$$2d+p+2+(d+p+2)^2,\ 2d+p+2+2(d+p+2)^2,\ 1).$$

The results of Theorem 2.5 apply to this system and its flow $\phi^{(1)}$. Note that $M_{s,t}$ is the predictable quadratic variation of the tensor product of $R_{s,t}$ with itself. Write $X^{(1)}_i$ for the coefficient vector fields of the w^i integrals in $\phi^{(1)}$, and $D^{(1)}_{s,t}$ for the Jacobian of $\phi^{(1)}$. Also write

$$R^{(1)}_{s,t} = \int_0^t (D^{(1)-1}_{s,u} X^{(1)}_i(u))^* \, \mathrm{d}w^i_u$$

and $M^{(1)}_{s,t}$ for the predictable quadratic variation of the tensor product of $R^{(1)}_{s,t}$ with $R^{(0)}_{s,t}$ which we shall denote by

$$M^{(1)}_{s,t} = \langle R^{(1)}_{s,t} \otimes R^{(0)}_{s,t} \rangle.$$

Then define $\phi^{(2)} = (\phi^{(1)}, D^{(1)}, R^{(1)}, M^{(1)})$ so $\phi^{(2)}$ is a Markov process for which the stochastic flow results of Theorem 2.5 hold. Proceeding in this way we inductively define $X^{(n)}_i$ for the coefficient vector fields of the w^i integrals in $\phi^{(n)}$.

$$R^{(n)}_{s,t} = \int_0^t (D^{(n)-1}_{s,u} X^{(n)}_i(u)) \, \mathrm{d}w^i_u,$$

$$M^{(n)}_{s,t} = \langle R^{(n)} \otimes R^{(0)} \rangle_{s,t}$$

and

$$\phi^{(n+1)} = (\phi^{(n)}, D^{(n)}, R^{(n)}, M^{(n)}).$$

Write ∇_n for the gradient operator in the components of $\phi^{(n)}$.

THEOREM 4.7. *Suppose c is a bounded C^∞ scalar function on R^d with bounded derivatives. Let g be a possibly vector, (or matrix), valued function on the state space of $\phi^{(n)}$ such that $g(\phi^{(n)}(0,t))$ and $\nabla_n g(\phi^{(n)}(0,t))$ are both in some $L^p(\Omega)$. Then*

$$\begin{aligned}
&E_h[c(x_{0,t})g(\phi^{(n)}(0,t))\otimes R_{0,T}^{(0)} \mid Y_T] \\
&\quad = E_h[c_x(x_{0,t})g(\phi^{(n)}(0,t))D_{0,t}M_{0,t} \mid Y_T] + \\
&\qquad + E_h[c(x_{0,t})\nabla_n g(\phi^{(n)}(0,t))D_{0,t}^{(n)}M_{0,t}^{(n)} \mid Y_T] + \\
&\qquad + E_h\left[L_{0,T}^{-1}c(x_{0,t})g(\phi^{(n)}(0,t))\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid Y_T\right] \\
&\qquad - E_h[c(x_{0,t})g(\phi^{(n)}(0,t))\Lambda_{0,T} \mid Y_T].
\end{aligned}$$

Proof. The result follows by applying to the system $\phi^{(n)}$ the techniques used to derive (4.11).

REMARKS 4.8. Theorem 2.5 implies

$$\sup_{s\le t}|D_{0,s}^{(n)}|,\quad \sup_{s\le t}|M_{0,s}^{(n)}|,\quad \sup_{s\le t}\left|\frac{\partial L_{0,s}}{\partial x_0}\right|,\quad \sup_{s\le t}|\Lambda_{0,s}|$$

are in $L^p(\Omega\times\tilde{\Omega}, P\times\tilde{P})$ for all $1\le p<\infty$ and, therefore, finite a.s. We have already noted that

$$\sup_{s\le t}|D_{0,s}^{-1}| \quad \text{and} \quad \sup_{s\le t} L_{0,s}^{-1}$$

are in every $L^p(\Omega\times\tilde{\Omega}, P\times\tilde{P})$, $1\le p<\infty$. To write out the above results in terms of $D_{0,t}$, $\partial L_{0,t}/\partial x$ and higher derivatives involves very involved calculations. Even in the one dimensional case, it seems better to introduce the sequence of flows $\phi^{(n)}$. Theorem 4.7 can again be thought of as giving a conditional 'integration by parts' formula for c_x.

COROLLARY 4.9. *If $M_{0,T}^{-1}$ is in some $L^p(\Omega\times\tilde{\Omega}, P_h)$ taking $g(\phi^{(1)}(0,t)) = M_{0,t}^{-1}D_{0,t}^{-1}$ in Theorem 4.7 we have*

$$\begin{aligned}
&E_h[c_x(x_{0,t}) \mid Y_T] \\
&\quad = E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\otimes R_{0,T} \mid Y_T] \\
&\qquad - E_h[c(x_{0,t})(\nabla_1 g)(D_{0,t}, M_{0,t})D_{0,t}^{(1)}M_{0,t}^{(1)} \mid Y_T] \\
&\qquad - E_h\left[c(x_{0,t})L_{0,T}^{-1}M_{0,t}^{-1}D_{0,t}^{-1}\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid Y_T\right] + \\
&\qquad + E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\Lambda_{0,t} \mid Y_T].
\end{aligned}$$

Because the remaining terms are integrable and, therefore, finite a.s., we have proved the following result:

THEOREM 4.10. *Suppose P_h is the probability measure of Definition 3.8 and $(x_{0,t}, y_{0,t})$ is the solution under P_h of (3.10). Let c be a smooth function with bounded derivatives of all orders. Then if $M_{0,T}^{-1}$ is in some $L^p(\Omega\times\tilde{\Omega}, P_h)$*

$$|E[c_x(x_{0,t}) \mid Y_T]| \leq K(y) \sup_{x\in R^d} |c(x)|, \tag{4.12}$$

where $K(y)$ is Y_T-measurable and finite a.s.

REMARKS 4.11. Condition (4.12) implies that the random variable $x_{0,t}(x_0, y_0)$ has a conditional density given Y_T, $d(x)$, $x \in R^d$ for almost all y. (See Malliavin [8] or Zakai [13].) Now for any $s \leq T$, $Y_s \subset Y_T$. So by Jensen's inequality, from (4.12)

$$|E[c_x(x_{0,t}) \mid Y_s]| \leq K'(y) \sup_{x\in R^d} |c(x)|. \tag{4.13}$$

Equation (4.13) holds for $s \leq t$ or $s \geq t$ so the prediction, filtering and smooothing problems for the random variable $x_{0,t}(x_0, y_0)$ all have a density for almost all y.

The remaining question concerns the existence and integrability properties of $M_{0,T}^{-1}$. These have been carefully studied, see Bismut [1], Malliavin [8] and Stroock [11]. For $(x, y) \in R^d \times R^p$ write $T_{x,y}$ for the vector subspace of R^d generated by the vector fields $\dot{X}_1(x, y), \ldots, X_m(x, y)$, and the Lie brackets of $X_1(x, y), \ldots, X_m(x, y)$ and $\tilde{X}_1(x, y), \ldots, \tilde{X}_n(x, y)$, where each bracket contains at least one of the vector fields $X_1(x, y), \ldots, X_m(x, y)$. Then in Theorem 1.19 of [2] Bismut and Michel show that for all $T > 0$, $M_{0,T}^{-1}$ is in $L^p(\Omega\times\hat{\Omega}, P_h)$ for all $1 \leq p < \infty$ if the following condition H, analogous to a condition of Hörmander is satisfied:

$$H: T_{x_0,y_0} \text{ is equal to the whole of } R^d.$$

As Bismut [1] observes, if H is satisfied at (x_0, y_0) then it is satisfied in some neighbourhood of (x_0, y_0).

Finally recall that if u is a nonsingular linear map of R^d to itself, then the map $\phi: u \to u^{-1}$ has a derivative $\phi'(u)$ which is a linear map on the space of linear maps of R^d to itself given by $\phi'(u)\cdot h = -u^{-1}\cdot h\cdot u^{-1}$. Applying this to $g(D_{0,t}, M_{0,t}) = M_{0,t}^{-1}D_{0,t}^{-1}$, we have

$$\begin{aligned}
&E_h[c_x(x_{0,t}) \mid Y_T] \\
&\quad = E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1} \otimes R_{0,T} \mid Y_T] \\
&\qquad - E_h[c(x_{0,t})M_{0,t}^{-1}((\nabla_1 M_{0,t})(D_{0,t}^{(1)}M_{0,t}^{(1)})M_{0,t}^{-1}D_{0,t}^{-1} \mid Y_T] \\
&\qquad - E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}((\nabla_1 D_{0,t})(D_{0,t}^{(1)}M_{0,t}^{(1)})D_{0,t}^{-1} \mid Y_T] \\
&\qquad - E_h\left[c(x_{0,t})L_{0,T}^{-1}M_{0,t}^{-1}D_{0,t}^{-1}\frac{\partial L_{0,T}}{\partial x_0}M_{0,t} \mid Y_T\right] \\
&\qquad + E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\Lambda_{0,T} \mid Y_T].
\end{aligned} \tag{4.14}$$

5. Bounds for Higher Derivatives

To show the conditional density of $x_{0,t}$ is differentiable, in the prediction, filtering and smoothing situations, we shall obtain bounds for higher derivatives of the form:

$$\left| E\left[\frac{\partial^\alpha c}{\partial x^\alpha}(x_{0,t}) \mid Y_T\right]\right| \leq K(y) \sup_{x\in R^d}|c(x)|, \tag{5.1}$$

where $0 < t \leq T$. Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index of nonnegative integers and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

Again, if $0 \leq s \leq T$, then Jensen's inequality applied to (5.1) gives

$$\left| E\left[\frac{\partial^\alpha c}{\partial x^\alpha}(x_{0,t}) \mid Y_s\right]\right| \leq K'(y) \sup_{x\in R^d}|c(x)|. \tag{5.2}$$

A well-known argument from harmonic analysis (see [10], or [12]) shows that if (5.2) is true for all α with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq n$ where $n \geq d+1$, then the random variable $x_{0,t}(x_0, y_0)$ has a conditional density $d(x)$ given Y_s, which is in $C^{n-d-1}(R^d)$. That is, we have a differentiable conditional density in the prediction, filtering and smoothing situations.

To see how to proceed apply Corollary 4.9 to c_x in place of c. (If preferred, Corollary 4.9 could be applied to just one partial derivative $\partial c/\partial x_k$ in place of c. However, the result is true for vector functions c.) This gives

$$\begin{aligned}
&E_h[c_{xx}(x_{0,t}) \mid Y_T] \\
&\quad = E_h[c_x(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1} \otimes R_{0,T} \mid Y_T] \\
&\qquad - E_h[c_x(x_{0,t})(\nabla_1 g)(D_{0,t}, M_{0,t})D_{0,t}^{(1)}M_{0,t}^{(1)} \mid Y_T] \\
&\qquad - E_h\left[c_x(x_{0,t})L_{0,T}^{-1}M_{0,t}^{-1}D_{0,t}^{-1}\frac{\partial L_{0,t}}{\partial x_0}M_{0,T} \mid Y_T\right] + \\
&\qquad + E_h[c_x(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\Lambda_{0,T} \mid Y_T].
\end{aligned} \tag{5.3}$$

Consider the four terms on the right of (5.3). Each term is of the form

$$E_h[c_x(x_{0,t})h_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T)) \mid Y_T], \quad i = 1,2,3,4.$$

For each such h_i consider a function $\tilde{h}_i = h_i M_{0,t}^{-1}D_{0,t}^{-1}$ and apply Theorem 4.7 to c and $\tilde{h}_i$ to obtain

$$\begin{aligned}
&E_h[c_x(x_{0,t})h_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T)) \mid Y_T] \\
&\quad = E_h[c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T)) \otimes R_{0,T}^{(0)} \mid Y_T] \\
&\qquad - E_h[c(x_{0,t})(\nabla_n(t)\tilde{h}_i)(\phi^{(1)}(0,t), \phi^{(1)}(0,T))D_{0,t}^{(2)}M_{0,t}^{(2)} \mid Y_T]
\end{aligned}$$

$$- E_h[c(x_{0,t})(\nabla_n(T)\tilde{h}_i)(\phi^{(1)}(0,t), \phi^{(1)}(0,T))D^{(2)}_{0,T}M^{(2)}_{0,T} \mid Y_T]$$

$$- E_h\left[L^{-1}_{0,T}c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T))\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} \mid Y_T\right]$$

$$+ E_h[c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T))\Lambda_{0,T} \mid Y_T].$$

Substituting in (5.3) we obtain an expression on the right which involves only c and not its derivatives. This procedure can be repeated, using Theorem 4.7. At each stage, to replace a term of the form

$$E_h[c_x(x_{0,t})h(\phi^{(n)}(0,t), \phi^{(n)}(0,T)) \mid Y_T]$$

by one involving only c define $\tilde{h} = hM^{-1}_{0,t}D^{-1}_{0,t}$ and applying Theorem 4.7. Clearly, higher powers of $M^{-1}_{0,t}$ are introduced at each iteration. However, Hörmander's condition H is sufficient to ensure that $M^{-1}_{0,t}$ is in every $L^p(\Omega \times \tilde{\Omega}, P_h)$, $1 \leq p < \infty$. We have, therefore, proved the following result:

THEOREM 5.1. *Suppose condition H is satisfied. Then the random variable* $x_{0,t}(x_0, y_0)$, *the solution of the signal process, has a conditional density given* Y_s *for almost all y which is in* $C^\infty(R^d)$ *for* $s \geq t$ *and* $s \leq t$. *That is, under condition H the prediction, filtering and smoothing problems have a smooth density solution.*

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A SHORT PROOF OF A MARTINGALE REPRESENTATION RESULT

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Abstract: Using the Ito differentiation rule, the properties of stochastic flows and the unique decomposition of special semimartingales, the integrand in a stochastic integral is quickly identified.

Keywords: martingale representation, Malliavin calculus.

Introduction

It is well known that a martingale with respect to the filtration generated by a Brownian motion is a stochastic integral of the Brownian motion. There have been many derivations of the integrand in this representation; see Clarke (1970/71), Haussmann (1979), Bismut (1981) and Davis (1980). In this note we give a very short derivation in the Markov case using the properties of stochastic flows and the unique decomposition of special semimartingales.

Flows

Suppose $w=(w^1,\dots,w^n)$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$. Consider the stochastic differential equation

$$\mathrm{d}x_t = f(t, x_t)\,\mathrm{d}t + \sigma(t, x_t)\,\mathrm{d}w_t \tag{1}$$

for $t \geq 0$, where $f:[0,\infty)\times R^n \to R^n$ and $\sigma:[0,\infty)\times R^n \to R^n \times R^n$ are measurable functions which are three times differentiable in $x \in R^n$. Write $\xi_{t,s}(x)$ for the solution of (1) for $s \geqslant t$, having initial condition $\xi_{t,t}(x)=x$. Then from the results of Bismut [1] there is a set $N \subset \Omega$ such that for $\omega \notin N$ there is a version of $\xi_{t,s}(x)$ which is twice differentiable in x and continuous in t and s.

Write $z_s = \partial \xi_{t,s}/\partial x$ for the Jacobian of the map $\xi_{t,s}$. Then it is known that z_s is the solution of the linearized equation

$$\mathrm{d}z_s = f_x(s, x_s)z_s\,\mathrm{d}s + \sigma_x(s, x_s)z_s\,\mathrm{d}w_s$$

with initial condition $z_t = I$, the $n \times n$ identity matrix.

Consider $0 \leqslant t \leqslant T$, an initial condition $x_0 \in R^n$ at time $t=0$ and a function $c(\xi_{0,T}(x_0))$ of the final position of the trajectory. Here c is a differentiable, real valued function on R^n such that $c(\xi_{0,T}(x_0))$ and $c_\xi(\xi_{0,T}(x_0))$ are integrable. Write $\{F_t\}$ for the right continuous complete family of σ-fields generated by $\sigma\{w_s : s \leqslant t\}$.

Then $M_t = E[c(\xi_{0,T}(x_0)) \mid F_t]$ is an $\{F_t\}$-martingale, and so by, for example, Theorem 12.33 of [4], M_t has a representation

$$M_t = M_0 + \int_0^t \gamma_s \, \mathrm{d}w_s \tag{2}$$

where γ is an $\{F_t\}$-predictable process.

Theorem

$$\begin{aligned}\gamma_s &= E_{s,x}\left[c_\xi(\xi_{0,T}(x_0))\frac{\partial \xi_{s,T}(x)}{\partial x}\right]\sigma(s, \xi_{0,s}(x_0)) \\ &= E\left[c_\xi(\xi_{0,T}(x_0))\frac{\partial \xi_{0,T}(x)}{\partial x} \mid F_s\right]\left(\frac{\partial \xi_{0,s}}{\partial x}\right)^{-1}\sigma(s, \xi_{0,s}(x_0)).\end{aligned}$$

Proof. For $0 \leqslant t \leqslant T$ write $x = \xi_{0,t}(x_0)$, so that, by the semigroup property of the stochastic flows,

$$\xi_{0,T}(x_0) = \xi_{t,T}(\xi_{0,t}(x_0)) = \xi_{t,T}(x).$$

Then

$$\begin{aligned}M_t &= E[c(\xi_{0,T}(x_0)) \mid F_t] = E[c(\xi_{t,T}(x)) \mid F_t] = E_{t,x}[c(\xi_{t,T}(x))] \\ &= V(t, x), \quad \text{say}.\end{aligned}$$

As noted above, $\xi_{t,T}(x)$ is twice differentiable in x. The differentiability of $E[c(\xi_{t,T}(x)) \mid F_t]$ in t can be seen by writing the backward equation for $\xi_{t,T}(x)$ as in Kunita [7]. However, $x = \xi_{0,t}(x_0)$ so expanding $V(t, \xi_{0,t}(x_0))$ by the Ito rule

$$\begin{aligned}V(t, \xi_{0,t}(x_0)) = M_t = V(0, x_0) &+ \int_0^t\left(\frac{\partial V}{\partial t}(s, \xi_{0,s}(x_0)) + LV(s, \xi_{0,s}(x_0))\right)\mathrm{d}s \\ &+ \int_0^t \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0))\sigma(s, \xi_{0,s}(x_0))\,\mathrm{d}w_s.\end{aligned} \tag{3}$$

Here,

$$L = \sum_{i=1}^n f^i \frac{\partial}{\partial x_i} + \tfrac{1}{2}\sum_{i,j=1}^n a^{ij}\frac{\partial^2}{\partial x_i \partial x_j} \quad \text{where } a(t, x_t) \text{ is the matrix } \sigma\sigma^*.$$

However,

$$\begin{aligned}\frac{\partial V(t, x)}{\partial x} &= E\left[\frac{\partial c}{\partial x}(\xi_{t,T}(x)) \mid F_t\right] = E\left[c_\xi(\xi_{t,T}(x))\frac{\partial \xi_{t,T}(x)}{\partial x} \mid F_t\right] \\ &= E_{t,x}\left[c_\xi(\xi_{0,T}(x_0))\frac{\partial \xi_{t,T}(x)}{\partial x}\right].\end{aligned}$$

M_t is certainly a special semimartingale, so the decompositions (2) and (3) must be the same.

In particular, equating the martingale terms we have

$$\begin{aligned}\gamma_s &= E_{s,x}\left[c_\xi(\xi_{0,T}(x_0))\frac{\partial \xi_{s,T}(x)}{\partial x}\right]\sigma(s, \xi_{0,s}(x_0)) \\ &= E\left[c_\xi(\xi_{0,T}(x_0))\frac{\partial \xi_{0,T}(x)}{\partial x} \mid F_s\right]\left(\frac{\partial \xi_{0,s}}{\partial x}\right)^{-1}\sigma(s, \xi_{0,s}(x_0)).\end{aligned}$$

Remarks

As there is no bounded variation term in (2) we must also have immediately:

$$\frac{\partial V}{\partial t}(s, \xi_{0,s}(x_0)) + LV(s, \xi_{0,s}(x_0)) = 0 \quad \text{with } V(0, x_0) = E\left[c(\xi_{0,T}(x_0))\right].$$

The techniques can be extended to more general martingales as in Davis (1980). Similar techniques quickly give the results of the Malliavin calculus in this situation.

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INTEGRATION BY PARTS, HOMOGENEOUS CHAOS EXPANSIONS AND SMOOTH DENSITIES[1]

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By iterating a martingale representation result a homogeneous chaos expansion is obtained. Using the martingale representation, the integration-by-parts formula of the Malliavin calculus is derived using properties of stochastic flows. The infinite-dimensional calculus of variations is not required.

1. Introduction. Since Malliavin's outstanding breakthrough [9] there have been other treatments and simplifications of the Malliavin calculus, including those of Bismut [2], Stroock [11], Bichteler and Fonken [1] and Norris [10]. In this paper we apply a very simple representation of the integrand in a stochastic integral, Theorem 3.1, to first derive the homogeneous chaos expansion of a certain random variable. An integration-by-parts formula is obtained and, if the Malliavin matrix M has an inverse which belongs to every $L^p(\Omega)$ (a condition guaranteed by Hörmander's H_1 hypothesis), it is shown the diffusion has a smooth density. The principle simplification in this paper is the observation that by considering an enlarged Markov system only the simple stochastic integral representation of Theorem 3.1 is needed. No infinite-dimensional calculus is required.

2. Flows. In this section we recall some definitions and properties of stochastic flows on d-dimensional Euclidean space. Suppose $w_t = (w_t^1, \dots, w_t^m)$, $0 \le t$, is an m-dimensional Brownian motion on (Ω, F, P). Write $\{F_t\}$ for the right-continuous complete filtration generated by w. Let $X_0, X_1, \dots, X_m$ be smooth vector fields on $[0, \infty) \times R^d$ all of whose derivatives are bounded. Then from Bismut [2] or Carverhill and Elworthy [4] we quote the following result.

THEOREM 2.1. *There is a map ξ: $\Omega \times [0, \infty) \times [0, \infty) \times R^d \to R^d$ such that:*
(i) *For $0 \le s \le t$ and $x \in R^d$ $\xi_{s,t}(x)$ is the essentially unique solution of the stochastic differential equation*

$$d\xi_{s,t}(x) = X_0\big(t, \xi_{s,t}(x)\big)\,dt + X_i\big(t, \xi_{s,t}(x)\big)\,dw_t^i, \tag{2.1}$$

with $\xi_{s,s}(x) = x$. (Note the Einstein summation convention is used.)

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(ii) *For each* ω, s, t *the map* $\xi_{s,t}(\cdot)$ *is* C^∞ *on* $R^d \to R^d$ *with a first derivative, the Jacobian,* $\partial \xi_{s,t}/\partial x = D_{s,t}$, *which satisfies*

$$(2.2)\qquad dD_{s,t} = \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}\,dt + \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}\,dw_t^i,$$

with initial condition $D_{s,s} = I$, *the* $d \times d$ *identity matrix.*

(iii) *If* $W_{s,t} = \partial^2 \xi_{s,t}/\partial x^2$ *is the second derivative, then*

$$(2.3)\qquad \begin{aligned} dW_{s,t} = {}& \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))W_{s,t}\,dt + \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))W_{s,t}\,dw_t^i \\ &+ \frac{\partial^2 X_0}{\partial \xi^2}(t, \xi_{s,t}(x))D_{s,t} \otimes D_{s,t}\,dt + \frac{\partial^2 X_i}{\partial \xi^2}(t, \xi_{s,t}(x))D_{s,t} \otimes D_{s,t}\,dw_t^i, \end{aligned}$$

with $W_{s,s} = 0 \in (R^d \otimes R^d) \otimes R^d$.

REMARKS 2.2. Note that (2.2) and (2.3) are obtained formally by differentiating (2.1). However, if we consider the enlarged stochastic system given by (2.1)–(2.3) for $(\xi_{s,t}, D_{s,t}, W_{s,t})$, the coefficients are not bounded. Nevertheless, Norris [10] has extended the results of Theorem 2.1. to such systems. To state Norris' results we first define a class of "lower triangular" coefficients.

DEFINITION 2.3. For positive integers $\alpha, d, d_1, \dots, d_k$ write $S_\alpha(d_1, \dots, d_k)$ for the set of $X \in C^\infty(R^d, R^d)$ of the form

$$(2.4)\qquad X(x) = \begin{pmatrix} X^{(1)}(x^1) \\ X^{(2)}(x^1, x^2) \\ \vdots \\ X^{(k)}(x^1, x^2, \dots, x^k) \end{pmatrix}, \quad \text{for } x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix},$$

where R^d is identified with $R^{d_1} \times \cdots \times R^{d_k}$, $x^j \in R^{d_j}$ and the X satisfy

$$(2.5)\qquad \|X\|_{S(\alpha, N)} = \sup_{x \in R^d}\left(\sup_{0 \le n \le N} \frac{|D^n X(x)|}{(1 + |x|^\alpha)} \vee \sup_{1 \le j \le k} \left|D_j X^{(j)}(x)\right| \right) < \infty,$$

for all positive integers N. Write $S(d_1, \dots, d_k) = \cup_\alpha S_\alpha(d_1, \dots, d_k)$.

REMARKS 2.4. Note (2.1)–(2.3) can be considered as a single system whose coefficients are not bounded but are in $S(d, d^2, d^3)$. The final supremum on the right of (2.5) implies the first derivatives of $X^{(1)}$ are bounded, as are the first derivatives D_j in the "new" variable x^j of $X^{(j)}(x^1, \dots, x^j)$. This means $X^{(j)}$ is allowed linear growth in x^j, a situation illustrated in (2.2) and (2.3). We quote from Norris [10] the following result.

THEOREM 2.5. *Let $X_0, X_1, \ldots, X_m \in S_\alpha(d_1, \ldots, d_k)$. Then there is a map $\phi: \Omega \times [0, \infty) \times [0, \infty) \times R^d \to R^d$ such that:*

(i) *For $0 \leq s \leq t$ and $x \in R^d$, $\phi(\omega, s, t, x)$ is the essentially unique solution of the stochastic differential equation*

$$dx_t = X_0(x_t)\,dt + X_i(x_t)\,dw_t^i, \tag{2.6}$$

with $x_s = x$.

(ii) *For each ω, s, t the map $\phi(\omega, s, t, x)$ is C^∞ in x with derivatives of all orders satisfying stochastic differential equations obtained from (2.6) by formal differentiation.*

$$\text{(iii)} \quad \sup_{|x| \leq R} E\left[\sup_{s \leq u \leq t} |D^n\phi(\omega, s, u, x)|^p\right] \tag{2.7}$$
$$\leq C(p, s, t, R, N, d_1, \ldots, d_k, \alpha, \|X_0\|_{S(\alpha, N)}, \ldots, \|X_m\|_{S(\alpha, N)}).$$

REMARKS 2.6. Norris proves Theorem 2.5. by induction on j. Write (2.6) as a system of stochastic differential equations for $j = 1, \ldots, k$,

$$\begin{aligned} dx_t^j &= X_0^{(j)}(x_t^1, \ldots, x_t^j)\,dt + X_i^{(j)}(x_t^1, \ldots, x_t^j)\,dw_t^i, \\ x_s^j &= x^j \in R^{d_j}. \end{aligned} \tag{2.8}$$

Suppose the result is true for $1, \ldots, j-1$ and write $\tilde{X}_i^{(j)}(\omega, s, t, x^j) = X_i^{(j)}(x_t^1(\omega), \ldots, x_t^{j-1}(\omega), x^j)$. Then (2.8) can be written in the form

$$dx_t^j = \tilde{X}_0(s, t, x_t^j)\,dt + \tilde{X}_i(s, t, x_t^j)\,dw_t^i$$

and Theorem (2.1) applied. The difficult step is establishing the result for $j = 1$. However, this follows using a stopping argument, a technique employed by Bismut [2, 3]. Using the notation of Theorem 2.1, the following result is well known.

LEMMA 2.7. *For $0 \leq s \leq t$ write $V_{s,t}$ for the solution of*

$$\begin{aligned} dV_{s,t} = &- V_{s,t}\left(\frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))\right) - \sum_{i=1}^m \left(\frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))^2\right) dt \\ &- V_{s,t}\frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))\,dw_t^i, \end{aligned} \tag{2.9}$$

with $V_{s,s} = I$. Then $D_{s,t}V_{s,t} = I$, the $d \times d$ identity matrix.

PROOF. Applying Itô's rule to $V_{s,t}D_{s,t}$, we see $d(V_{s,t}D_{s,t}) = 0$. However, $V_{s,s}D_{s,s} = I$. □

REMARKS 2.8. An application of Jensen's, Burkholder's and Gronwall's inequalities shows that $\sup_{s \le u \le t} |D_{s,u}|$, $\sup_{s \le u \le t} |W_{s,u}|$ and $\sup_{s \le u \le t} |V_{s,u}|$ are in $L^p(\Omega)$ for all $p < \infty$. Alternatively, this conclusion follows from applying Theorem 2.5 to the system (2.1)–(2.3) and (2.9). For $0 \le s \le t$, by the uniqueness of the solution of (2.1)

$$\begin{aligned} \xi_{0,t}(x_0) &= \xi_{s,t}(\xi_{0,s}(x_0)) \\ &= \xi_{s,t}(x), \quad \text{if } x = \xi_{0,s}(x_0). \end{aligned} \tag{2.10}$$

Differentiating (2.10), using the chain rule,

$$D_{0,t} = D_{s,t} D_{0,s} \tag{2.11}$$

and

$$W_{0,t} = W_{s,t}(D_{0,s} \otimes D_{0,s}) + D_{s,t} W_{0,s}. \tag{2.12}$$

3. Representation and series expansion. Suppose $0 \le t \le T$ and $\xi_{0,t}(x_0)$ is the solution of the stochastic differential equation (2.1). Consider a real-valued twice continuously differentiable function c for which the random variable $c(\xi_{0,T}(x_0))$ and the components of the gradient $c_\xi(\xi_{0,T}(x_0))$ are integrable. Let M_t be the right-continuous version of the martingale

$$E\left[c(\xi_{0,T}(x_0)) | F_t\right].$$

We then have the following representation result.

THEOREM 3.1. *For* $0 \le t \le T$, $M_t = E[c(\xi_{0,T}(x_0))] + \int_0^t \gamma_i(s)\, dw_s^i$, *where*

$$\gamma_i(s) = E\left[c_\xi(\xi_{0,T}(x_0)) D_{0,T} | F_s\right] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)).$$

PROOF. It is well known (see [5], for example) that any F_t-martingale M_t has a representation

$$M_t = M_0 + \int_0^t \gamma_i(s)\, dw_s^i, \tag{3.1}$$

for some predictable integrands γ_i. Because the process $\xi_{0,t}(x_0)$ is Markov

$$\begin{aligned} M_t &= E\left[c(\xi_{0,T}(x_0)) | F_t\right] \\ &= E\left[c(\xi_{t,T}(x)) | F_t\right] \\ &= E_{t,x}\left[c(\xi_{t,T}(x))\right] \\ &= V(t,x), \quad \text{say, where } x = \xi_{0,t}(x_0). \end{aligned} \tag{3.2}$$

By the chain rule and Theorem 2.1, $c(\xi_{t,T}(x))$ is differentiable, in fact smooth, in x. The differentiability of $E[c(\xi_{t,T}(x)) | F_t]$ in t can be established by writing the backward equation for $\xi_{t,T}(x)$ as in Kunita [8]. Consequently, applying the Itô

rule to $V(t, x)$, with $x = \xi_{0,t}(x_0)$,

$$V\big(t, \xi_{0,t}(x_0)\big) = V(0, x_0) + \int_0^t \left(\frac{\partial V}{\partial s} + LV\right) ds + \int_0^t \frac{\partial V}{\partial x}\big(s, \xi_{0,s}(x_0)\big) X_i\big(s, \xi_{0,s}(x_0)\big)\, dw_s^i, \tag{3.3}$$

where

$$L = \sum_{i=1}^{d} X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\sum_{k=1}^{m} X_k^i X_k^j \right) \frac{\partial^2}{\partial x_i\, \partial x_j}.$$

By the uniqueness of the decomposition of special semimartingales, comparing (3.1) and (3.3), we must have (as is well known)

$$\frac{\partial V}{\partial s} + LV = 0,$$

and

$$\gamma_i(s) = \frac{\partial V}{\partial x}\big(s, \xi_{0,s}(x_0)\big) X_i\big(s, \xi_{0,s}(x_0)\big).$$

From (3.2)

$$\frac{\partial V}{\partial x} = E\big[c_\xi\big(\xi_{s,T}(x)\big) D_{s,T} | F_s\big]$$

so by (2.11)

$$\gamma_i(s) = E\big[c_\xi\big(\xi_{0,T}(x_0)\big) D_{0,T} | F_s\big] D_{0,s}^{-1} X_i\big(s, \xi_{0,s}(x_0)\big). \qquad \square$$

REMARKS 3.2. Note in particular the representation

$$c\big(\xi_{0,T}(x_0)\big) = E\big[c\big(\xi_{0,T}(x_0)\big)\big] + \int_0^T E\big[c_\xi\big(\xi_{0,T}(x_0)\big) D_{0,T} | F_s\big] D_{0,s}^{-1} X_i\big(s, \xi_{0,s}(x_0)\big)\, dw_s^i. \tag{3.4}$$

Theorem 3.1 can be extended immediately to vector (or matrix) functions c. Finally, it seems the proof of Theorem 3.1 can be extended to the non-Markov case ([6]).

3.3 NOTATION. Write $\xi^{(0)} = \xi$ for the solution flow of Theorem 2.1, and $D^{(0)} = D$ for its Jacobian given by (2.2). Write $\xi^{(1)}$ for the $d + d^2$-dimensional process with components $\xi^{(1)} = (\xi^{(0)}, D^{(0)})$. Write $D^{(1)}$ for the Jacobian of this $d + d^2$-dimensional process. Write $\xi^{(2)}$ for the process $\xi^{(2)} = (\xi^{(1)}, D^{(1)})$ and so on. Then $\xi^{(n+1)} = (\xi^{(n)}, D^{(n)})$. Note $\xi^{(n)}$ is a process for which the stochastic flow

results of Theorem 2.5 hold. Write

$$c^{(1)}\big(\xi^{(1)}_{0,T}(x_0)\big) = \frac{\partial c}{\partial \xi}\big(\xi_{0,T}(x_0)\big) D_{0,T},$$

$$c^{(2)}\big(\xi^{(2)}_{0,T}(x^{(2)}_0)\big) = \frac{\partial c^{(1)}}{\partial \xi^{(1)}}\big(\xi^{(1)}_{0,T}(x^{(1)}_0)\big) D^{(1)}_{0,T}$$

and so on, so

$$c^{(n+1)}\big(\xi^{(n+1)}_{0,T}(x^{(n+1)}_0)\big) = \frac{\partial c^{(n)}}{\partial \xi^{(n)}}\big(\xi^{(n)}_{0,T}(x^{(n)}_0)\big) D^{(n)}_{0,T}.$$

Note the initial condition at 0 for the variable $D^{(n)}$ is always the identity matrix of appropriate dimension. Write $X^{(n)}_i$ for the vector field coefficient of w^i in the stochastic differential equation defining $\xi^{(n)}$ and abbreviate

$$X^{(n)}_i\big(s, \xi^{(n)}_{0,s}(x^{(n)}_0)\big) \quad \text{as } X^{(n)}_i(s).$$

Then by iterating Theorem 3.1, we have the following representation of the random variable $c(\xi_{0,T}(x_0))$.

THEOREM 3.4. *If c has bounded derivatives of all order, then for any n,*

$$\begin{aligned} c(\xi_{0,T}(x_0)) &= E\big[c(\xi_{0,T}(x_0))\big] + \sum_{k=1}^{n} E\big[c^{(k)}\big(\xi^{(k)}_{0,T}(x^{(k)}_0)\big)\big] \\ &\quad \times \int_0^T\Big(\int_0^{s_1}\cdots\Big(\int_0^{s_n} D^{(n-1)-1}_{0,s_n} X_k(s_n)\, dw^k_{s_n}\Big)\cdots\Big) D^{-1}_{0,s} X_i(s)\, dw^i_s \\ &\quad + \int_0^T\Big(\int_0^{s_1}\cdots\Big(\int_0^{s_n} E\big[c^{(n+1)}|F_{s_{n+1}}\big] D^{(n)-1}_{0,s_{n+1}} X_j(s_{n+1})\, dw^j_{s_{n+1}}\Big)\cdots\Big) \\ &\quad \times D^{-1}_{0,s} X_i(s)\, dw^i_s. \end{aligned} \tag{3.5}$$

PROOF. From (3.4)

$$\begin{aligned} c\big(\xi_{0,T}(x_0)\big) &= E\big[c\big(\xi_{0,T}(x_0)\big)\big] + \int_0^T E\big[c^{(1)}|F_s\big] D^{-1}_{0,s} X_i(s)\, dw^i_s \\ &= E[c] + E\big[c^{(1)}\big]\int_0^T D^{-1}_{0,s} X_i(s)\, dw^i_s \\ &\quad + \int_0^T\Big(\int_0^{s_1} E\big[c^{(2)}|F_{s_1}\big] D^{(1)-1}_{0,s_1} X^{(1)}_i(s_1)\, dw^i_{s_1}\Big) D^{-1}_{0,s} X_j(s)\, dw^j_s. \end{aligned}$$

The result follows by repeated application of the representation of Theorem 3.1. □

REMARKS 3.5. In principle, it is possible to write the previous expansion in terms of $D_{0,t}$, $W_{0,t}$ and higher derivatives of the diffeomorphism $\xi_{0,t}$, rather than by considering higher and higher dimensional systems. However, this gives rise to very complicated formulae. Consider the case when $d = 1$. Then $\xi^{(1)} = (\xi^{(0)}, D^{(0)})$ is two dimensional and by Theorem 3.1

$$
\begin{aligned}
E\left[c^{(1)}(\xi^{(1)})|F_s\right] &= E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right] \\
&\quad + \int_0^s E\left[c_{\xi\xi}(\xi_{u,T}(x))D_{0,T}D_{u,T}\right. \\
&\qquad \left. + c_\xi(\xi_{u,T}(x))W_{u,T}D_{0,u}|F_u\right] X_\iota(u)\, dw_u^\iota \qquad (3.6) \\
&\quad + \int_0^s E\left[c_\xi(\xi_{u,T}(x_0))D_{u,T}|F_u\right] \frac{\partial X_\iota}{\partial \xi}(u)\, dw_u^\iota.
\end{aligned}
$$

Here we are writing $\xi_{0,T}(x_0) = \xi_{u,T}(x)$, where $x = \xi_{0,u}(x_0)$, and $D_{0,T} = D_{u,T}D$, where $D = D_{0,u}$. Note the final integral in (3.6) is a result of differentiating in the D variables. Recalling (2.11) and (2.12), we have

$$
\begin{aligned}
E\left[c^{(1)}(\xi^{(1)})|F_s\right] &= E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right] \\
&\quad + \int_0^s E\left[c_{\xi\xi}(\xi_{0,T}(x_0))D_{0,T}^2\right. \\
&\qquad \left. + c_\xi(\xi_{0,T}(x_0))W_{0,T}|F_u\right] D_{0,u}^{-1}X_\iota(u)\, dw_u^\iota \\
&\quad - \int_0^s E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_u\right] D_{0,u}^{-2}W_{0,u}X_\iota(u)\, dw_u^\iota \qquad (3.7) \\
&\quad + \int_0^s E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_u\right] D_{0,u}^{-1} \frac{\partial X_\iota}{\partial \xi}(u)\, dw_u^\iota \\
&= E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right] + \int_0^s \gamma_j(u,2)\, dw_u^j,
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_j(u,2) &= \alpha(u,2,1)D_{0,u}^{-1}X_j(u) - \alpha(u,2,2)D_{0,u}^{-2}W_{0,u}X_j(u) \\
&\quad + \alpha(u,2,2)D_{0,u}^{-1}\frac{\partial X_j}{\partial \xi}(u). \qquad (3.8)
\end{aligned}
$$

Here

$$\alpha(u,2,1) = E\left[c_{\xi\xi}(\xi_{0,T}(x_0))D_{0,T}^2 + c_\xi(\xi_{0,t}(x_0))W_{0,T}|F_u\right]$$

and

$$\alpha(u,2,2) = E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_u\right].$$

Substituting (3.7) in (3.4), we have

$$
\begin{aligned}
c(\xi_{0,T}(x_0)) &= E\left[c(\xi_{0,t}(x_0))\right] + E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right]\int_0^T D_{0,s}^{-1}X_i(s)\,dw_s^i \\
&\quad + \int_0^T\left(\int_0^s \gamma_j(u,2)\,dw_u^j\right)D_{0,s}^{-1}X_i(s)\,dw_s^i.
\end{aligned}
\tag{3.9}
$$

In turn, the martingales $\alpha(u,2,1)$ and $\alpha(u,2,2)$ can be expressed as stochastic integrals. Substituting again, we have

$$
\begin{aligned}
c(\xi_{0,T}(x_0)) &= E\left[c(\xi_{0,T}(x_0))\right] + E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right]\int_0^T D_{0,s}^{-1}X_i(s)\,dw_s^i \\
&\quad + E\left[c_{\xi\xi}(\xi_{0,T}(x_0))D_{0,T}^2 + c_\xi(\xi_{0,t}(x_0))W_{0,T}\right] \\
&\qquad \times \int_0^T\left(\int_0^s D_{0,u}^{-1}X_j(u)\,dw_u^j\right)D_{0,s}^{-1}X_i(s)\,dw_s^i \\
&\quad - E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right]\int_0^T\left(\int_0^s D_{0,u}^{-2}W_{0,u}X_j(u)\,dw_u^j\right)D_{0,s}^{-1}X_i(s)\,dw_s^i \\
&\quad + E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\right]\int_0^T\left(\int_0^s D_{0,u}^{-1}\frac{\partial X_j}{\partial\xi}(u)\,dw_u^j\right)D_{0,s}^{-1}X_i(s)\,dw_s^i \\
&\quad + \int_0^T\left\{\int_0^s\left(\int_0^u \gamma_k(v,3)\,dw_v^k\right)D_{0,s}^{-1}X_j(u)\,dw_s^j\right. \\
&\quad + \int_0^s\left(\int_0^u \gamma_k(v,4)\,dw_v^k\right)D_{0,s}^{-2}W_{0,u}X_j(u)\,dw_s^j \\
&\quad \left. + \int_0^s\left(\int_0^u \gamma_k(v,5)\,dw_v^k\right)D_{0,s}^{-1}\frac{\partial X_j(u)}{\partial\xi}\,dw_u^j\right\}D_{0,s}^{-1}X_i(s)\,dw_s^i.
\end{aligned}
\tag{3.10}
$$

REMARKS 3.6. Theorem 3.4 [or (3.10) in the one-dimensional case] indicates how a "Taylor series" expansion for the random variable $c(\xi_{0,T}(x_0))$ can be obtained as the sum of multiple stochastic integrals.

The coefficients of the stochastic integrals are functions of the expected values of $c(\xi_{0,T}(x_0))$ and its derivatives, and the Jacobian $D_{0,T}$ and its derivatives. The integrands in the multiple stochastic integrals do not involve c, but are functions of the Jacobian and its derivatives, and the coefficient functions X_i. By uniqueness the expansion is the same as the homogeneous chaos representation. This expansion can be used to investigate variations about the expected trajectory and large deviation problems ([7]).

COROLLARY 3.7. *Taking* $c(\xi_{0,T}(x_0)) = \xi_{0,T}(x_0) \in R^d$, *so* $c_\xi = I_d$, *the* $d \times d$ *identity matrix, and* $c_{\xi\xi} = 0$, (3.4) *gives*

$$
\xi_{0,T}(x_0) = E\left[\xi_{0,T}(x_0)\right] + \int_0^T\left[D_{0,T}|F_s\right]D_{0,s}^{-1}X_i(s)\,dw_s^i,
$$

with corresponding higher-order expansions.

LEMMA 3.8. *Write * to denote the transpose. Suppose c and g are real-valued, differentiable functions such that the random variables $c(\xi_{0,T}(x_0))$, $g(\xi_{0,T}(x_0))$, $c_\xi(\xi_{0,T}(x_0))$, $g_\xi(\xi_{0,T}(x_0))$ are in $L^2(\Omega)$. Then*

$$\begin{aligned} E\left[c(\xi_{0,T}(x_0))g(\xi_{0,T}(x_0))\right] &= E\left[c(\xi_{0,T}(x_0))\right]E\left[g(\xi_{0,T}(x_0))\right] \\ &\quad + E\left[\sum_{i=1}^{m}\int_0^T E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s\right]\right. \\ &\qquad \left.\times D_{0,s}^{-1}X_i(s)X_i^*(s)D_{0,s}^{*-1}E\left[g_\xi^*(\xi_{0,T}(x_0))|F_s\right]ds\right]. \end{aligned}$$

PROOF. By Theorem 3.1

$$g(\xi_{0,T}(x_0)) = E\left[g(\xi_{0,T}(x_0))\right] + \int_0^T E\left[g_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s\right]D_{0,s}^{-1}X_i(s)\,dw_s^i.$$

The result follows by taking the expectation of the product with (3.4). (Note $g^* = g$.) □

DEFINITION 3.9. The nonnegative matrix

$$M_{s,t} = \sum_{i=1}^{m}\int_s^t D_{s,u}^{-1}X_i(u)X_i^*(u)D_{s,u}^{*}\,du$$

will be called the Malliavin matrix for the system (2.1). Note that something similar to $M_{0,T}$ appears in Lemma 3.8. In some references, [11] and [12], the matrix $D_{0,T}M_{0,T}D_{0,T}^*$ is called the Malliavin matrix.

4. Integration by parts.

THEOREM 4.1. *Suppose c is a twice continuously differentiable scalar function such that $c(\xi_{0,T}(x_0))$ and $c_\xi(\xi_{0,T}(x_0))$ are square integrable. Then for any square-integrable predictable process $u(s) = (u_1(s), \ldots, u_m(s))$,*

$$\begin{aligned} E&\left[c(\xi_{0,T}(x_0))\int_0^T u_i(s)\,dw_s^i\right] \\ &= \sum_{i=1}^{m}E\left[c_\xi(\xi_{0,t}(x_0))D_{0,T}\int_0^T D_{0,s}^{-1}X_i(s)u_i(s)\,ds\right]. \end{aligned}$$

PROOF. Using the representation (3.4),

$$\begin{aligned} E&\left[c(\xi_{0,T}(x_0))\int_0^T u_i(s)\,dw_s^i\right] \\ &= \sum_{i=1}^{m}E\left[\int_0^T E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s\right]D_{0,s}^{-1}X_i(s)u_i(s)\,ds\right] \end{aligned}$$

and by Fubini's theorem this is

$$= \sum_{i=1}^{m} E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\int_0^T D_{0,s}^{-1}X_i(s)u_i(s)\,ds\right]. \qquad \square$$

COROLLARY 4.2. *The result is still true for vector- (or matrix-) valued functions c.*

COROLLARY 4.3. *Taking each $u_i(s)$ to be $(D_{0,s}^{-1}X_i(s))^*$, we have*

$$E\left[c(\xi_{0,T}(x_0))\int_0^T(D_{0,s}^{-1}X_i(s))^*\,dw_s^i\right] = E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}M_{0,T}\right].$$

COROLLARY 4.4. *Consider a product function*

$$h(\xi_{0,T}(x_0)) = c(\xi_{0,T}(x_0))g(\xi_{0,T}(x_0))$$

satisfying the conditions of the theorem. Then

$$\begin{aligned} &E\left[c(\xi_{0,T}(x_0))g(\xi_{0,t}(x_0))\int_0^T(D_{0,s}^{-1}X_i(s))^*\,dw_s^i\right] \\ (4.1)\qquad &= E\big[\big(c_\xi(\xi_{0,T}(x_0))g(\xi_{0,t}(x_0)) \\ &\qquad + c(\xi_{0,T}(x_0))g_\xi(\xi_{0,T}(x_0))\big)D_{0,T}M_{0,T}\big]. \end{aligned}$$

REMARKS 4.5. What we would like to do in (4.1) is take

$$g = M_{0,T}^{-1}D_{0,T}^{-1},$$

so that we can obtain a bound for c_ξ. However, $D_{0,T}^{-1}$ and $M_{0,T}^{-1}$ involve the past of the processes $\xi_{0,T}$, $D_{0,T}$, $M_{0,T}$. This difficulty can be circumvented by considering an enlarged system, similar to the technique used in Section 3. However, the sequence of enlarged systems is different to that discussed in Section 3, so different notation will be used. Note that even when the original process ξ is one dimensional the method leads to a discussion of higher-dimensional processes, so not much simplification is obtained by taking $d = 1$.

4.6 NOTATION. Write $\phi^{(0)}(\omega, s, t, x) = \xi_{s,t}(x)$ for the stochastic flow defined by (2.1). Now $D_{s,t}^{(0)}(x) = D_{s,t}(x)$ denotes the Jacobian of the flow $\phi^{(0)}$. From (2.11), if $D = D_{0,s}$ and $x = \xi_{0,s}(x_0)$,

$$D_{0,t}^{(0)}(x_0) = D_{s,t}(x)D,$$

so the system $(\phi^{(0)}, D^{(0)})$ is Markov. Write $R_{s,t}^{(0)}(x) = \int_s^t(D_{s,u}^{-1}X_i(u))^*\,dw_u^i$ and $R = R_{0,s}^{(0)}$. Then $R_{0,t}^{(0)} = R + D^{-1}R_{s,t}^{(0)}(x)$, so the system $(\phi^{(0)}, D^{(0)}, R^{(0)})$ is Markov. Finally, recall the definition (3.9) of $M_{s,t}$ and write $M_{s,t}^{(0)} = M_{s,t}$, $M = M_{0,s}^{(0)}$. Then $M_{0,t}^{(0)} = M + D^{-1}M_{s,t}(x)D^{*-1}$ and the system

$$\phi^{(1)} = (\phi^{(0)}, D^{(0)}, R^{(0)}, M^{(0)})$$

is Markov with coefficients in

$$S(d, d + d^2, 2d + d^2, 2d + d^2).$$

Consequently, Theorem 2.5 applies to this system and its stochastic flow $\phi^{(1)}$. Note that $M_{s,t}$ is the predictable quadratic variation of the tensor product of $R_{s,t}$ with $R^*_{s,t}$. Write $X_i^{(1)}$ for the coefficient vector fields of w^i in $\phi^{(1)}$. Furthermore, write $D_{s,t}^{(1)}$ for the Jacobian of $\phi^{(1)}$, $R_{s,t}^{(1)} = \int_s^t (D_{s,u}^{(1)-1} X_i^{(1)}(u))^* \, dw_u^i$ and $M_{s,t}^{(1)}$ for the predictable quadratic variation of the tensor product of $R_{s,t}^{(1)}$ with $R_{s,t}^{(0)*}$, which we shall denote by

$$M_{s,t}^{(1)} = \langle R_{s,t}^{(1)} \otimes R_{s,t}^{(0)} \rangle.$$

Then define

$$\phi^{(2)} = \left(\phi^{(1)}, D^{(1)}, R^{(1)}, M^{(1)}\right),$$

so $\phi^{(2)}$ is a Markov process for which the results of Theorem 2.5 hold. Proceeding in this way, we inductively define $\phi^{(n+1)} = (\phi^{(n)}, D^{(n)}, R^{(n)}, M^{(n)})$, where $R^{(n)} = \int_s^t (D_{s,u}^{(n)-1} X_i^{(n)}(u)) \, dw_u^i$ and $M^{(n)} = \langle R^{(n)} \otimes R^{(0)} \rangle$. Write ∇_n for the gradient operator in the components of $\phi^{(n)}$.

THEOREM 4.7. *Suppose c is a bounded C^∞ scalar function on R^d with bounded derivatives. Let g be a possibly vector- (or matrix-) valued function on the state space of $\phi^{(n)}$ such that $g(\phi^{(n)}(0, T, x_0))$ and $\nabla_n g(\phi^{(n)}(0, T, x_0))$ are both in some $L^p(\Omega)$. Then*

$$\begin{aligned} E&\left[c\left(\phi^{(0)}(0,T)\right)g\left(\phi^{(n)}(0,T)\right) \otimes R_{0,T}^{(0)}\right] \\ &= E\left[(\nabla_0 c)\left(\phi^{(0)}(0,T)\right)g\left(\phi^{(n)}(0,T)\right)D_{0,T}M_{0,T}\right] \\ &\quad + E\left[c\left(\phi^{(0)}(0,T)\right)(\nabla_n g)\left(\phi^{(n)}(0,T)\right)D_{0,T}^{(n)}M_{0,T}^{(n)}\right]. \end{aligned}$$

PROOF. Applying Theorem 3.1 to cg, we have

$$\begin{aligned} c&\left(\phi^{(0)}(0,T)\right)g\left(\phi^{(n)}(0,T)\right) \\ &= E\left[c\left(\phi^{(0)}(0,T)\right)g\left(\phi^{(n)}(0,T)\right)\right] \\ &\quad + \int_0^T E\left[(\nabla_0 c)\left(\phi^{(0)}(0,T)\right)g\left(\phi^{(n)}(0,T)\right)D_{0,T} | F_s\right] D_{0,s}^{-1} X_i(s) \, dw_s^i \\ &\quad + \int_0^T E\left[c\left(\phi^{(0)}(0,T)\right)(\nabla_n g)\left(\phi^{(n)}(0,T)\right)D_{0,T}^{(n)} | F_s\right] D_{0,s}^{(n)-1} X_i^{(n)}(s) \, dw_s^i. \end{aligned}$$

Taking the tensor product with $R_{0,T}^{(0)*}$ and the expected value, the result follows. □

REMARKS 4.8. To write out the preceding result in terms of $D_{0,t}$, $W_{0,t}$ and higher derivatives of the flow involves very involved calculations. Even in dimension 1 it seems better to introduce the sequence $\phi^{(n)}$ of flows. Note Theorem 2.5 implies $\sup_{s \le t} |D_{0,s}^{(n)}|, \sup_{s \le t} |M_{0,s}^{(n)}|$ are in every $L^p(\Omega)$. Theorem 4.7 is an integration by parts formula as only one term involves the gradient of

derivatives $\nabla_0 c = c_\xi$ of c.

COROLLARY 4.9. *Taking* $g(\phi^{(1)}(0,T)) = M_{0,T}^{-1}D_{0,T}^{-1}$, *if* $M_{0,T}^{-1}$ *is in some* $L^p(\Omega)$,

$$
\begin{aligned}
E\left[c_\xi\left(\xi_{0,T}(x_0)\right)\right] &= E\left[c\left(\xi_{0,t}(x_0)\right)M_{0,T}^{-1}D_{0,T}^{-1} \otimes R_{0,T}^{(0)}\right] \\
&\quad - E\left[c\left(\xi_{0,T}(x_0)\right)(\nabla_1 g)\left(D_{0,T}, M_{0,t}\right)D_{0,T}^{(1)}M_{0,T}^{(1)}\right].
\end{aligned} \tag{4.2}
$$

Because the remaining terms are integrable we have, therefore, proved the following result.

THEOREM 4.10. *Suppose* $\xi_{0,t}(x_0)$ *is the solution of* (2.1) *and* c *is a bounded smooth function with bounded derivatives. Then if* $M_{0,T}^{-1}$ *is in some* $L^p(\Omega)$,

$$
\left|E\left[c_\xi\left(\xi_{0,T}(x_0)\right)\right]\right| \le K \sup_{x \in R^d} |c(x)|. \tag{4.3}
$$

REMARKS 4.11. It is well known that inequality (4.3) implies that the random variable $\xi_{0,T}(x_0)$ has a density (see Malliavin [9] or Stroock [11]). The remaining question concerns the existence and integrability properties of $M_{0,T}^{-1}$. These have been carefully studied (see Malliavin [9], Stroock [11] and Norris [10]). In fact, it is known that $M_{0,T}^{-1}$ is in $L^p(\Omega)$ for all $p < \infty$ if the following condition H_1 of Hörmander on the coefficient vector fields $X_0, \dots, X_m$ of (2.1) is satisfied.

CONDITION H_1. $X_1, \dots, X_m, [X_i, X_j]$, for $i, j = 0, \dots, m$, $[X_i, [X_j, X_k]]$ for $i, j, k = 0, \dots, m$, etc. evaluated at $x_0 \in R^d$ span R^d.

Finally, recall that, if u is a nonsingular linear map of R^d to itself, then the map ϕ: $u \to u^{-1}$ has a derivative $\phi'(u)$, which is a linear map on the space of linear maps of R^d to itself, given by $\phi'(u)h = -u^{-1}hu^{-1}$. Applying this to $g(D_{0,T}, M_{0,T}) = M_{0,T}^{-1}D_{0,T}^{-1}$, we have

$$
\begin{aligned}
&E\left[c_\xi\left(\xi_{0,T}(x_0)\right)\right] \\
&\quad = E\left[c\left(\xi_{0,T}(x_0)\right)M_{0,T}^{-1}D_{0,T}^{-1} \otimes R_{0,T}^{(0)}\right] \\
&\qquad + E\left[c\left(\xi_{0,T}(x_0)\right)M_{0,T}^{-1}\left(\left(\nabla_1 M_{0,T}\right)\left(D_{0,T}^{(1)}M_{0,T}^{(1)}\right)\right)M_{0,T}^{-1}D_{0,T}^{-1}\right] \\
&\qquad + E\left[c\left(\xi_{0,T}(x_0)\right)M_{0,T}^{-1}D_{0,T}^{-1}\left(\left(\nabla_1 D_{0,T}\right)\left(D_{0,T}^{(1)}M_{0,T}^{(1)}\right)\right)D_{0,T}^{-1}\right].
\end{aligned} \tag{4.4}
$$

5. Bounds for higher derivatives. To show the density of $\xi_{0,T}(x_0)$ is differentiable, we must obtain bounds for higher derivatives of the form

$$
\left|E\left[\frac{\partial^\alpha c}{\partial \xi^\alpha}\left(\xi_{0,T}(x_0)\right)\right]\right| \le K \sup_{x \in R^d} |c(x)|. \tag{5.1}
$$

Here $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex of nonnegative integers and

$$\frac{\partial^\alpha}{\partial \xi^\alpha} = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \xi_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial \xi_d^{\alpha_d}}.$$

In fact, a well-known argument from harmonic analysis (see [10]) implies that if (5.1) is true for all α with $|\alpha| = \alpha_1 + \cdots + \alpha_d \le n$, where $n \ge d+1$, then the random variable $\xi_{0,T}(x_0)$ has a density $d(x) = d(x_1, \dots, x_d)$ which is in $C^{n-d-1}(R^d)$.

To see how to proceed, apply (4.2) to c_ξ in place of c. [If preferred, (4.2) could be applied to just one partial derivative $\partial c/\partial \xi_k$ in place of c; however, the result of Corollary 4.9 is true for vector functions c.] This gives

$$\begin{aligned} E\left[c_{\xi\xi}(\xi_{0,T}(x_0))\right] &= E\left[c_\xi(\xi_{0,T}(x_0)) M_{0,T}^{-1} D_{0,T}^{-1} \otimes R_{0,T}^{(0)}\right] \\ &\quad - E\left[c_\xi(\xi_{0,T}(x_0))(\nabla_1 g)(D_{0,T}, M_{0,T}) D_{0,T}^{(1)} M_{0,T}^{(1)}\right]. \end{aligned} \tag{5.2}$$

Consider the two terms on the right,

$$E\left[c_\xi(\xi_{0,T}(x_0)) M_{0,T}^{-1} D_{0,T}^{-1} \otimes R_{0,T}\right] \tag{5.3}$$

and

$$E\left[c_\xi(\xi_{0,T}(x_0))(\nabla_1 g)(D_{0,T}, M_{0,T}) D_{0,T}^{(1)} M_{0,T}^{(1)}\right]. \tag{5.4}$$

5.1 NOTATION. Write $M = M_{0,T}$, $D = D_{0,T}$, $D^{(1)} = D_{0,T}^{(1)}$, etc. Let $g_1(\phi^{(1)})$ be the function $M^{-1}D^{-1} \otimes RM^{-1}D^{-1}$ and $g_2(\phi^{(2)})$ be the function $(\nabla_1 g)(D, M) D^{(1)} M^{(1)} M^{-1} D^{-1}$.

Applying Theorem 4.7 to c and g_1, we have

$$\begin{aligned} &E\left[c(\xi_{0,T}(x_0)) g_1(\phi^{(1)}) \otimes R_{0,T}^{(0)}\right] \\ &\quad = E\left[c_\xi(\xi_{0,T}(x_0)) M_{0,T}^{-1} D_{0,T}^{-1} \otimes R_{0,T}^{(0)}\right] \\ &\qquad + E\left[c(\xi_{0,T}(x_0))(\nabla_2 g_1)(\phi^{(2)}(0,T)) D_{0,T}^{(2)} M_{0,T}^{(2)}\right]. \end{aligned} \tag{5.5}$$

Applying Theorem 4.7 to c and g_2 we have

$$\begin{aligned} &E\left[c(\xi_{0,T}(x_0)) g_2(\phi^{(2)}) \otimes R_{0,T}\right] \\ &\quad = E\left[c_\xi(\xi_{0,T}(x_0))(\nabla_1 g)(D_{0,T} M_{0,T}) D_{0,t}^{(1)} M_{0,T}^{(1)}\right] \\ &\qquad + E\left[c(\xi_{0,T}(x_0))(\nabla_3 g_2)(\phi^{(3)}(0,T)) D_{0,T}^{(3)} M_{0,T}^{(3)}\right]. \end{aligned} \tag{5.6}$$

Substituting in (5.2), we obtain an expression on the right which involves only c and not its derivatives. This procedure can be iterated, using Theorem 4.7. At any stage, to replace a term of the form $E[c_\xi(\xi_{0,T}(x_0)) h(\phi^{(n)}(0,T))]$ by one involving only c define $\tilde{h}(\phi^{(n)}(0,T)) = h(\phi^{(n)}(0,T)) M_{0,T}^{-1} D_{0,T}^{-1}$ and apply Theorem 4.7. Clearly, higher powers of $M_{0,T}^{-1}$ are introduced at each iteration. [From

Theorem 2.5 $D_{0,T}^{-1}$ is in every $L^p(\Omega)$.] Hörmander's condition H_1 is sufficient to ensure that $M_{0,T}^{-1}$ is in every $L^p(\Omega)$, $1 \le p < \infty$. We have, therefore, proved the following result.

THEOREM 5.2. *Suppose Hörmander's condition* H_1 *is satisfied. Then the random variable* $\xi_{0,T}(x_0)$ *has a density* $d(x)$ *which is in* $C^\infty(R^d)$.

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003

THE PARTIALLY OBSERVED STOCHASTIC MINIMUM PRINCIPLE*

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Abstract. Using stochastic flows and the generalized differentiation formula of Bismut and Kunita, the change in cost due to a strong variation of an optimal control is explicitly calculated. Differentiating this expression gives a minimum principle in both the partially observed and stochastic open loop situations. In the latter case the equation satisfied by the adjoint process is obtained by applying a martingale representation result.

Key words. stochastic control, minimum principle, adjoint process, stochastic flow

AMS(MOS) subject classification. 93E20

1. Introduction. Various proofs have been given of the minimum principle satisfied by an optimal control in a partially observed stochastic control problem. See, for example, the papers by Bensoussan [1], Elliott [8], Haussmann [11], and the recent paper [14] by Haussmann in which the adjoint process is identified. The simple case of a partially observed Markov chain is discussed in the University of Maryland lecture notes [9] of Elliott.

In this article we show that the minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. The results of Bismut [5], [6] and Kunita [16] on stochastic flows enable us to compute in an easy and explicit way the change in the cost due to a "strong variation" of an optimal control. The only technical difficulty is the justification of the differentiation. As we wished to exhibit the simplification obtained by using the ideas of stochastic flows, the result is not proved under the weakest possible hypotheses. In § 6, stochastic open loop controls are considered and a similar minimum principle with an explicit adjoint process is derived in § 7. If the optimal control is Markov, the equation satisfied by the adjoint process is obtained in § 8 using the martingale representation result of [10]. This simplifies the proof of Haussmann [12]. Finally in § 9 it is pointed out how Bensoussan's minimum principle [2] follows from our result if the drift coefficient is differentiable in the control variable.

2. Dynamics. Suppose the state of the system is described by a stochastic differential equation

$$\begin{aligned} d\xi_t &= f(t, \xi_t, u)\,dt + g(t, \xi_t)\,dw_t, \\ \xi_t &\in R^d, \quad \xi_0 = x_0, \quad 0 \leqq t \leqq T. \end{aligned} \tag{2.1}$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. We shall make the following assumptions:

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(A_1) x_0 is given; if x_0 is a random variable and P_0 its distribution, the situation when $\int |x|^q P_0(dx) < \infty$ for some $q > n+1$ can be treated, as in [14], by including an extra integration with respect to P_0.

(A_2) $f:[0,T]\times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x and for some constant K_1, $(1+|x|)^{-1}|f(t,x,u)|+|f_x(t,x,u)| \leqq K_1$.

(A_3) $g:[0,T]\times R^d \to R^d \otimes R^n$ is a matrix-valued function, Borel measurable, continuously differentiable in x, and for some constant K_2, $|g(t,x)|+|g_x(t,x)| \leqq K_2$.

The observation process is given by

$$dy_t = h(\xi_t)\,dt + dv_t, \qquad y_t \in R^m, \quad y_0 = 0, \quad 0 \leqq t \leqq T. \tag{2.2}$$

In the above equations $w=(w^1,\cdots,w^n)$ and $v=(v^1,\cdots,v^d)$ are independent Brownian motions. We also assume the following:

(A_4) $h: R^d \to R^m$ is Borel measurable, continuously differentiable in x, and for some constant K_3, $|h(t,x)|+|h_x(t,x)| \leqq K_3$.

Remark 2.1. These hypotheses can be weakened. For example, in (A_4), h can be allowed linear growth in x. Because g is bounded, a delicate argument then implies the exponential Z of (2.3) is in some L^p space, $1<p<\infty$. (See, for example, Theorem 2.2 of [13].) However, when h is bounded, Z is in all the L^p spaces (see Lemma 2.3). Also, if we require f to have linear growth in u, then the set of control values U can be unbounded as in [14]. Our objective, however, is not the greatest generality but is to demonstrate the simplicity of the techniques of stochastic flows.

Let $\hat{P}$ denote Wiener measure on $C([0,T],R^n)$ and μ denote Wiener measure on $C([0,T],R^m)$. Consider the space $\Omega = C([0,T],R^n)\times C([0,T],R^m)$ with coordinate functions (w_t, y_t) and define Wiener measure P on Ω by

$$P(dw,dy) = \hat{P}(dw)\mu(dy).$$

DEFINITION 2.2. Write $Y=\{Y_t\}$ for the right continuous complete filtration on $C([0,T],R^m)$ generated by $Y_t^0 = \sigma\{y_s : s \leqq t\}$. The set of admissible control functions $\underline{U}$ will be the Y-predictable functions on $[0,T]\times C([0,T],R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$ write $\xi^u_{s,t}(x)$ for the strong solution of (2.1) corresponding to control u, and with $\xi^u_{s,s}(x)=x$. Write

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))'\,dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2\,dr\right) \tag{2.3}$$

and define a new probability measure P^u on Ω by $dP^u/dP = Z^u_{0,T}(x_0)$. Then under P^u, $(\xi^u_{0,t}(x_0), y_t)$ is a solution of (2.1) and (2.2), that is, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion v such that y_t satisfies (2.2). A version of Z defined for every trajectory y of the observation process is obtained by integrating by parts the stochastic integral in (2.3).

LEMMA 2.3. *Under hypothesis* (A_4) *for* $t \leqq T$,

$$E[(Z^u_{0,t}(x_0))^p] < \infty \quad \textit{for all } u \in \underline{U} \textit{ and all } p, \quad 1 \leqq p < \infty.$$

Proof.

$$Z^u_{0,t}(x_0) = 1 + \int_0^t Z^u_{0,r}(x_0) h(\xi^u_{0,r}(x_0))'\,dy_r.$$

Therefore, for any p there is a constant C_p such that

$$E[(Z^u_{0,t}(x_0))^p] \leqq C_p\left[1+E\left(\int_0^t (Z^u_{0,r}(x_0))^2 h(\xi^u_{0,r}(x_0))^2\,dr\right)^{p/2}\right].$$

The result follows by Gronwall's inequality.

Cost 2.4. We shall suppose the cost is purely terminal and given by some bounded, continuously differentiable function

$$c(\xi^u_{0,T}(x_0)),$$

which has bounded derivatives. Then the expected cost, if control $u \in \underline{U}$ is used, is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

In terms of P, under which y_t is always a Brownian motion, this is

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.4}$$

3. Stochastic flows. For $u \in \underline{U}$ write

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)\,dr + \int_s^t g(r, \xi^u_{s,r}(x))\,dw_r \tag{3.1}$$

for the solution of (2.1) over the time interval $[s, t]$ with initial condition $\xi^u_{s,s}(x) = x$. In the sequel we wish to discuss the behavior of (3.1) for each trajectory y of the observation process. We have already noted that there is a version of Z defined for every y. The results of Bismut [5] and Kunita [16] extend easily and show the map

$$\xi^u_{s,t}: R^d \to R^d$$

is, almost surely, for each $y \in C([0, T], R^m)$ a diffeomorphism. Bismut [5] initially gives proofs when the coefficients f and g are bounded, but points out that a stopping time argument extends the results to when, for example, the coefficients have linear growth.

Write $\|\xi^u(x_0)\|_t = \sup_{0\leqq s\leqq t}|\xi^u_{0,s}(x_0)|$. Then, as in Lemma 2.1 of [13], for any p, $1 \leqq p < \infty$, using Gronwall's and Jensen's inequalities,

$$\|\xi^u(x_0)\|^p_T \leqq C\left(1+|x_0|^p + \left|\int_0^t g(r, \xi^u_{0,r}(x_0))\,dw_r\right|^p\right)$$

almost surely, for some constant C.

Therefore, using Burkholder's inequality and hypothesis (A_3), $\|\xi^u(x_0)\|_T$ is in L^p for all p, $1 \leqq p < \infty$.

Suppose $u^* \in \underline{U}$ is an optimal control; then $J(u^*) \leqq J(u)$ for any other $u \in \underline{U}$. Write $\xi^*_{s,t}(\cdot)$ for $\xi^{u^*}_{s,t}(\cdot)$. The derivative $\partial\xi^*_{s,t}(x)/\partial x$ is the matrix solution C_t of the equation for $s \leqq t$,

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t\,dt + \sum_{i=1}^n g^{(i)}_x(t, \xi^*_{s,t}(x))C_t\,dw^i_t \quad \text{with } C_s = I. \tag{3.2}$$

Here I is the $n \times n$ identity matrix and $g^{(i)}$ is the ith column of g. From hypotheses (A_2) and (A_3), f_x and g_x are bounded. When we write $\|C\|_t = \sup_{0\leqq s\leqq t}|C_s|$, an application of Gronwall's, Jensen's, and Burkholder's inequalities again implies $\|C\|_T$ is in

L^p for all p, $1 \leqq p < \infty$. Consider the related matrix-valued stochastic differential equation

$$
\begin{aligned}
D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r)'\, dr - \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x))'\, dw^i_r \\
+ \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x))')^2\, dr.
\end{aligned}
\tag{3.3}
$$

Then it can be checked that $D_t C_t = I$ for $t \geqq s$, so that D_t is the inverse of the Jacobian, that is, $D_t = (\partial \xi^*_{s,t}(x)/\partial x)^{-1}$. Again, because f_x and g_x are bounded we have that $\|D\|_t$ is in every L^p, $1 \leqq p < \infty$.

For a d-dimensional semimartingale z_t Bismut [5] shows that $\xi^*_{s,t}(z_t)$ is well-defined and gives the semimartingale representation of this process. In fact if $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i\, dw^i_r$ is a d-dimensional semimartingale, Bismut's formula states that

$$
\begin{aligned}
\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r, \xi^*_{s,r}(z_r), u^*_r) + \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r) \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \\
+ \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 \xi^*_{s,r}(z_r)}{\partial x^2}(H_i, H_i) \Big)\, dr \\
+ \int_s^t \frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\, dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big)\, dw^i_r.
\end{aligned}
\tag{3.4}
$$

DEFINITION 3.1. We shall consider perturbations of the optimal control u^* of the following kind. For $s \in [0, T)$, $h > 0$ such that $0 \leqq s < s + h \leqq T$, for any other admissible control $\tilde{u} \in \underline{U}$ and $A \in Y$, define a strong variation of u^* by

$$
u(t, w) = \begin{cases} u^*(t, w) & \text{if } (t, w) \notin [s, s+h] \times A, \\ \tilde{u}(t, w) & \text{if } (t, w) \in [s, s+h] \times A. \end{cases}
$$

Applying (3.4) as in Theorem 5.1 of [7], we have the following result.

THEOREM 3.2. *For the perturbation u of the optimal control u^* consider the process*

$$
z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))\, dr. \tag{3.5}
$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

Proof. Note that the equation defining z_t involves only an integral in time; there is no martingale term, so to apply (3.4) we have $H_i = 0$ for all i. Therefore, from (3.4)

$$
\begin{aligned}
\xi^*_{s,t}(z_t) = x + \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)\, dr \\
+ \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right) \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x} \right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))\, dr \\
+ \int_s^t g(r, \xi^*_{s,r}(z_r))\, dw_r.
\end{aligned}
$$

However, the solution of (3.1) is unique so

$$
\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x).
$$

Remark 3.3. Note that the perturbation $u(t)$ equals $u^*(t)$ if $t > s + h$ so $z_t = z_{s+h}$ if $t > s + h$ and

$$
\xi^*_{s,t}(z_t) = \xi^*_{s,t}(z_{s+h}) = \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)).
$$

4. Augmented flows. Consider the augmented flow that includes as an extra coordinate the stochastic exponential $Z^*_{s,t}$ with a "variable" initial condition $z \in R$ for $Z^*_{s,s}(\cdot)$. That is, consider the $(d+1)$-dimensional system given by

$$\xi^*_{s,t}(x) = x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)\, dr + \int_s^t g(r, \xi^*_{s,r}(x))\, dw_r,$$

$$Z^*_{s,t}(x, z) = z + \int_s^t Z^*_{s,r}(x, z) h(\xi^*_{s,r}(x))'\, dy_r.$$

Therefore, from the first equation in the proof of Lemma 2.3 we have

$$\begin{aligned} Z^*_{s,t}(x, z) &= z Z^*_{s,t}(x) \\ &= z \exp\left(\int_s^t h(\xi^*_{s,r}(x))'\, dy_r - \frac{1}{2}\int_s^t h(\xi^*_{s,r}(x))^2\, dr\right) \end{aligned}$$

and we see there is a version of the enlarged system defined for each trajectory y by integrating by parts the stochastic integral. The augmented map $(x, z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x, z))$ is then almost surely a diffeomorphism of R^{d+1}. Note that $\partial \xi^*_{s,t}(x)/\partial z = 0$, $\partial f/\partial z = 0$ and $\partial g/\partial z = 0$. The Jacobian of this augmented map is, therefore, represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \partial \xi^*_{s,t}(x)/\partial x & 0 \\ \partial Z^*_{s,t}(x, z)/\partial x & \partial Z^*_{s,t}(x, z)/\partial z \end{pmatrix},$$

and for $1 \leqq i \leqq d$ as in (3.2)

$$\begin{aligned} \frac{\partial Z^*_{s,t}(x, z)}{\partial x_i} = \sum_{j=1}^m \int_s^t \bigg(& Z^*_{s,r}(x, z) \frac{\partial h^j(\xi^*_{s,r}(x))}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}(x)}{\partial x_i} \\ & + h^j(\xi^*_{s,r}(x)) \frac{d Z^*_{s,r}(x, z)}{\partial x_i} \bigg)\, dy^j_r. \end{aligned} \tag{4.1}$$

(Here the double index k is summed from 1 to n.)

We shall be interested in the solution of this differential system (4.1) only in the situation when $z = 1$, so we shall write $Z^*_{s,t}(x)$ for $Z^*_{s,t}(x, 1)$. The following result is motivated by formally differentiating the exponential formula for $Z^*_{s,t}(x)$.

LEMMA 4.1.

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = Z^*_{s,t}(x)\left(\int_s^t h_x(\xi^*_{s,r}(x)) \cdot \frac{\partial \xi^*_{s,r}(x)}{\partial x} \cdot dv_r\right)$$

where $v = (v^1, \cdots, v^n)$ is the Brownian motion in the observation process.

Proof. From (4.1) we see $\partial Z^*_{s,t}(x)/\partial x$ is the solution of the stochastic differential equation

$$\frac{\partial Z^*_{s,t}(x)}{\partial x} = \int_s^t \left(\frac{\partial Z^*_{s,r}(x)}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}(x)}{\partial x}\right) dy_r. \tag{4.2}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\left(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} \cdot dv_r\right)$$

where

$$dy_r = h(\xi^*_{s,t}(x))\, dt + dv_t.$$

Because

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r$$

the product rule gives

$$L_{s,t}(x) = \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x}\, dv_r + \int_s^t \left(\int_s^r h_x \cdot \frac{\partial \xi^*_{s,\sigma}}{\partial x} \cdot dv_\sigma \right) Z^*_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r$$

$$+ \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) \cdot h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x}\, dr$$

$$= \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x))\, dy_r + \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} \cdot dy_r.$$

Therefore, $L_{s,t}(x)$ is also a solution of (4.2), so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^*_{s,t}(x)}{\partial x}.$$

Remark 4.2. As noted at the beginning of this section we can consider the augmented flow

$$(x, z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x, z)) \quad \text{for } x \in R^d, z \in R,$$

and we are only interested in the situation when $z = 1$, so we write $Z^*_{s,t}(x)$.

LEMMA 4.3. *$Z^*_{s,t}(z_t) = Z^u_{s,t}(x)$ where z_t is the semimartingale defined in* (3.6).

Proof. $Z^u_{s,t}(x)$ is the process uniquely defined by

$$Z^u_{s,t}(x) = 1 + \int_s^t Z^u_{s,r}(x) h'(\xi^u_{s,r}(x))\, dy_r. \tag{4.3}$$

Consider an augmented $(d+1)$-dimensional version of (3.5) defining a semimartingale $\bar{z}_t = (z_t, 1)$, so the additional component is always identically one. Then applying (3.4) to the new component of the augmented process, we have

$$Z^*_{s,r}(z_r) = 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^*_{s,r}(z_r))\, dy_r$$

$$= 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^u_{s,r}(x))\, dy_r$$

by Theorem 3.2. However, (4.3) has a unique solution so $Z^*_{s,t}(z_t) = Z^u_{s,t}(x)$.

Remark 4.4. Note that for $t > s + h$

$$Z^*_{s,t}(z_t) = Z^*_{s,t}(z_{s+h}).$$

5. The minimum principle. Control u will be the perturbation of the optimal control u^* as in Definition 3.1. We shall write $x = \xi^*_{0,s}(x_0)$. Then the minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0) c(\xi^*_{0,T}(x_0))]$$

$$= E[Z^*_{0,s}(x_0) Z^*_{s,T}(x) c(\xi^*_{s,T}(x))].$$

The cost corresponding to the perturbed control u is

$$J(u) = E[Z^*_{0,s}(x_0) Z^u_{s,T}(x) c(\xi^u_{s,T}(x))]$$

$$= E[Z^*_{0,s}(x_0) Z^*_{s,T}(z_{s+h}) c(\xi^*_{s,T}(z_{s+h}))]$$

by Theorem 3.2 and Lemma 4.3. Now $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are almost surely differentiable with continuous derivatives and z_t, given by (3.5), is absolutely continuous. Therefore,

$$J(u)-J(u^*)=E[Z^*_{0,s}(x_0)(Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))-Z^*_{s,T}(x)c(\xi^*_{s,T}(x)))]$$
$$=E\left[\int_s^{s+h}\Gamma(s,z_r)(f(r,\xi^*_{s,r}(z_r),u^*_r)-f(r,\xi^*_{s,r}(x),u^*_r))\,dr\right]$$

where by Lemma 4.1

$$\Gamma(s,z_r)=Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\left\{c_\xi(\xi^*_{s,T}(z_r))\frac{\partial\xi^*_{s,T}(z_r)}{\partial x}\right.$$
$$\left.+c(\xi^*_{s,T}(z_r))\left(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\frac{\partial\xi^*_{s,\sigma}}{\partial x}(z_r)\,dv_\sigma\right)\right\}\left(\frac{\partial\xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}.$$

Note that this expression gives an explicit formula for the change in the cost resulting from a variation in the optimal control. The only remaining problem is to justify differentiating the right-hand side.

From Lemma 2.3, Z is in every L^p space, $1\leqq p<\infty$, and from the remarks at the beginning of § 3, $C_T=\partial\xi^*_{s,T}/\partial x$ and $D_T=(\partial\xi^*_{s,T}/\partial x)^{-1}$ are in every L^p space, $1\leqq p<\infty$. Consequently, Γ is in every L^p space, $1\leqq p<\infty$.

Therefore,

$$J(u)-J(u^*)=\int_s^{s+h}E[(\Gamma(s,z_r)-\Gamma(s,x))(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[(\Gamma(s,x)-\Gamma(r,x))(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[\Gamma(r,x)(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r)$$
$$-f(r,\xi^*_{s,r}(x),u_r)+f(r,\xi^*_{s,r}(x),u^*_r))]\,dr$$
$$+\int_s^{s+h}E[\Gamma(r,x)(f(r,\xi^*_{0,r}(x_0),u_r)-f(r,\xi^*_{0,r}(x_0),u^*_r))]\,dr$$
$$=I_1(h)+I_2(h)+I_3(h)+I_4(h),\quad\text{say.}$$

Now,

$$|I_1(h)|\leqq K_1\int_s^{s+h}E[|\Gamma(s,z_r)-\Gamma(s,x)|(1+\|\xi^u(x_0)\|_{s+h})]\,dr$$
$$\leqq K_1h\sup_{s\leqq r\leqq s+h}E[|\Gamma(s,z_r)-\Gamma(s,x)|(1+\|\xi^u(x_0)\|_{s+h})],$$
$$|I_2(h)|\leqq K_2\int_s^{s+h}E[|\Gamma(s,x)-\Gamma(r,s)|(1+\|\xi^u(x_0)\|_{s+h})]\,dr$$
$$\leqq K_2h\sup_{s\leqq r\leqq s+h}E[|\Gamma(s,x)-\Gamma(r,x)|(1+\|\xi^u(x_0)\|_{s+h})],$$
$$|I_3(h)|\leqq K_3\int_s^{s+h}E[|\Gamma(r,x)|\,\|\xi^*_{s,r}(z_r)-\xi^*_{s,r}(x)\|]\,dr$$
$$\leqq K_3h\sup_{s\leqq r\leqq s+h}E[|\Gamma(r,x)|\,\|\xi^u_{s,\cdot}(x)-\xi^*_{s,\cdot}(x)\|_{s+h}].$$

The differences $|\Gamma(s, z_r) - \Gamma(s, x)|$, $|\Gamma(s, x) - \Gamma(r, x)|$ and $\|\xi^u_{s,\cdot}(x) - \xi^*_{s,\cdot}(x)\|_{s+h}$ are all uniformly bounded in some L^p, $p \geqq 1$, and

$$\lim_{r\to s} |\Gamma(s, z_r) - \Gamma(s, x)| = 0 \quad \text{a.s.},$$

$$\lim_{r\to s} |\Gamma(s, x) - \Gamma(r, x)| = 0 \quad \text{a.s.},$$

$$\lim_{h\to 0} \|\xi^u_{s,\cdot}(x) - \xi^*_{s,\cdot}(x)\|_{s+h} = 0.$$

Therefore,

$$\lim_{r\to s} \|\Gamma(s, z_r) - \Gamma(s, x)\|_p = 0,$$

$$\lim_{r\to s} \|\Gamma(s, x) - \Gamma(r, x)\|_p = 0, \quad \text{and}$$

$$\lim_{h\to 0} \|(\|\xi^u_{s,\cdot}(x) - \xi^*_{s,\cdot}(x)\|_{s+h})\|_p = 0 \quad \text{for some } p.$$

Consequently, $\lim_{h\to 0} h^{-1} I_k(h) = 0$, for $k = 1, 2, 3$.

The only remaining problem concerns the differentiability of

$$I_4(h) = \int_s^{s+h} E[\Gamma(r, x)(f(r, \xi^*_{0,r}(x_0), u_r) - f(r, \xi^*_{0,r}(x_0), u^*_r))]\, dr.$$

The integrand is almost surely in $L^1([0, T])$ so $\lim_{h\to 0} h^{-1} I_4(h)$ exists for almost every $s \in [0, T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the control u. Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

(1) Because the trajectories are, almost surely, continuous, Y_ρ is countably generated by sets $\{A_{i\rho}\}$, $i = 1, 2, \cdots$ for any rational number $\rho \in [0, T]$. Consequently, Y_t is countably generated by the sets $\{A_{i\rho}\}$, $\rho \leqq t$.

(2) Let G_t denote the set of measurable functions from (Ω, Y_t) to $U \subset R^k$. (If $u \in \underline{U}$ then $u(t, w) \in G_t$.) Using the L^1-norm, as in [8], there is a countable dense subset $H_\rho = \{u_{j\rho}\}$ of G_ρ, for rational $\rho \in [0, T]$. If $H_t = \bigcup_{\rho \leqq t} H_\rho$ then H_t is a countable dense subset of G_t. If $u_{j\rho} \in H_\rho$ then, as a function constant in time, $u_{j\rho}$ can be considered as an admissible control over the time interval $[t, T]$ for $t \geqq \rho$.

(3) The countable family of perturbations is obtained by considering sets $A_{i\rho} \in Y_t$, functions $u_{j\rho} \in H_t$, where $\rho \leqq t$, and defining as in (3.1) the following:

$$u^*_{j\rho}(s, w) = \begin{cases} u^*(s, w) & \text{if } (s, w) \notin [t, T] \times A_{i\rho}, \\ u_{j\rho}(s, w) & \text{if } (s, w) \in [t, T] \times A_{i\rho}. \end{cases}$$

Then for each i, j, ρ

$$\text{(5.1)} \qquad \lim_{h\to 0} h^{-1} \int_s^{s+h} E[\Gamma(r, x)(f(r, \xi^*_{0,r}(x_0), u^*_{j\rho}) - f(r, \xi^*_{0,r}(x_0), u^*))]\, dr$$

exists and equals

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*)) I_{A_{i\rho}}]$$

for almost all $s \in [0, T]$. Therefore, considering this perturbation we have

$$\lim_{h\to 0} h^{-1}(J(u^*_{j\rho}) - J(u^*)) = E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*)) I_{A_{i\rho}}]$$

$$\geqq 0 \quad \text{for almost all } s \in [0, T].$$

Consequently there is a set $S \subset [0, T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (5.1) exists for all i, j, ρ, and gives

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u_{j\rho}) - f(s, \xi^*_{0,s}(x_0), u^*))I_{A_{i\rho}}] \geqq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underline{U}$, we can deduce that if $s \notin S$, then

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u) - f(s, \xi^*_{0,s}(x_0), u^*))I_A] \geqq 0 \quad \text{for any } A \in Y_s. \tag{5.2}$$

Write

$$p_s(x) = E^*\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}(x)}{\partial x} + c(\xi^*_{0,T}(x_0))\left(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\frac{\partial \xi^*_{s,\sigma}(x)}{\partial x}\,dv_\sigma\right)\,\middle|\, Y_s \vee \{x\}\right]$$

where, as before, $x = \xi^*_{0,s}(x_0)$ and E^* denotes expectation under $P^* = P^{u^*}$. Then $p_s(x)$ is the co-state variable and we have in (5.2) proved the following "conditional" minimum principle.

THEOREM 5.1. *If $u^* \in \underline{U}$ is an optimal control there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$*

$$E^*[p_s(x)f(s, x, u^*) \mid Y_s] \geqq E^*[p_s(x)f(s, x, u) \mid Y_s] \quad a.s.$$

That is, the optimal control u^* almost surely minimizes the conditional Hamiltonian and the adjoint variable is $p_s(x)$.

6. Stochastic open loop controls. We shall again suppose the state of the system is described by a stochastic differential equation

$$d\xi_t = f(t, \xi_t, u)\,dt + g(t, \xi_t)\,dw_t, \qquad \xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leqq t \leqq T \tag{6.1}$$

where x_0, f, and g satisfy the same assumptions A_1, A_2, and A_3 as in § 2.

Suppose $w = (w^1, \cdots, w^n)$ is an n-dimensional Brownian motion on a probability space (Ω, F, P), with a right continuous complete filtration $\{F_t\}$, $0 \leqq t \leqq T$. Rather than controls depending on some observation process y we now consider controls that depend on the "noise process" w. These are sometimes called "stochastic open loop" controls [4].

DEFINITION 6.1. The set of admissible controls $\underline{V}$ will be the F_t-predictable functions on $[0, T] \times \Omega$ with values in a compact subset V of some Euclidean space R^k.

Remark 6.2. For each $u \in \underline{V}$ there is, therefore, a strong solution of (6.1) and we shall write $\xi^u_{s,t}(x)$ for the solution trajectory given by

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)\,dr + \int_s^t g(r, \xi^u_{s,r}(x))\,dw_r. \tag{6.2}$$

Again, because u is a (predictable) parameter the results of [2], [5], or [16] extend to this situation, so the derivative $\partial \xi^u_{s,t}/\partial x(x) = C^u_{s,t}$ exists and is the solution of

$$C^u_{s,t} = I + \int_s^t f_\xi(r, \xi^u_{s,r}(x), u_r)C^u_{s,r}\,dr + \sum_{k=1}^n \int_s^t g^{(k)}_\xi(r, \xi^u_{s,r}(x))C^u_{s,r}\,dw^k_r. \tag{6.3}$$

Suppose $D^u_{s,t}$ is the matrix-valued process defined by

$$\begin{aligned} D^u_{s,r} = I &- \int_s^t D^u_{s,t}\left(f_\xi(r, \xi^u_{s,r}(x), u_r) - \sum_{k=1}^n g^{(k)}_\xi(r, \xi^u_{s,r}(x))^2\right)dr \\ &- \sum_{k=1}^n \int_s^t D^u_{s,r} g^{(k)}_\xi(r, \xi^u_{s,r}(x))\,dw^k_r. \end{aligned} \tag{6.4}$$

Using the Itô rule as in § 3 we see that $d(D^u_{s,t}C^u_{s,t})=0$ and $D^u_{s,s}C^u_{s,s}=I$, so

$$D^u_{s,t}=(C^u_{s,t})^{-1}.$$

As before, if

$$\|\xi^u(x_0)\|_t=\sup_{0\leqq s\leqq t}|\xi^u_{0,s}(x_0)|,$$

$$\|C^u\|_T=\sup_{0\leqq s\leqq T}|C^u_{0,s}|,\qquad \|D^u\|_T=\sup_{0\leqq s\leqq T}|D^u_{0,s}|,$$

then applications of Gronwall's, Jensen's, and Burkholder's inequalities imply that

$$\|\xi^u(x_0)\|_k,\quad \|C^u\|_T,\quad\text{and}\quad \|D^u\|_T$$

are in L^p for all p, $1\leqq p<\infty$.

Cost 6.3. As in § 2, we shall suppose the cost is purely terminal and given by a bounded C^2 function

$$c(\xi^u_{0,T}(x_0)).$$

Furthermore, we shall assume

$$|c(x)|+|c_x(x)|+|c_{xx}(x)|\leqq K_3(1+|x|^q)$$

for some $q<\infty$.

The expected cost if a control $u\in \underline{V}$ is used, therefore, is

$$J(u)=E[c(\xi^u_{0,T}(x_0))].$$

Suppose there is an optimal control $u^*\in\underline{V}$ so that

$$J(u^*)\leqq J(u)\quad\text{for all } u\in\underline{V}.$$

Notation 6.4. If u^* is an optimal control, write ξ^* for ξ^{u^*}, C^* for C^{u^*}, etc.

DEFINITION 6.5. Consider perturbations of u^* of the following kind. For $s\in[0,T]$, $h>0$ such that $0\leqq s<s+h\leqq T$ and $A\in F_s$ define, for any other $\tilde u\in\underline{V}$, a strong variation of u^* by

$$u(t,w)=\begin{cases}u^*(t,w) & \text{if } (t,w)\notin[s,s+h]\times A,\\ \tilde u(t,w) & \text{if } (t,w)\in[s,s+h]\times A.\end{cases}$$

The following result is established exactly as Theorem 3.2.

THEOREM 6.6. *For any perturbation u of u^* consider the process*

$$z_t=x+\int_s^t\left(\frac{\partial\xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}(f(r,\xi^*_{s,r}(z_r),u_r)-f(r,\xi^*_{s,r}(z_r),u^*_r))\,dr.\tag{6.5}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

Note if $t>s+h$, $\xi^*_{s,t}(z_t)=\xi^*_{s,t}(z_{s+h})=\xi^*_{s+h,t}(\xi^u_{s,s+h}(x))$.

7. An open loop minimum principle. Now

$$\begin{aligned}J(u^*)&=E[c(\xi^*_{0,T}(x_0))]\\&=E[c(\xi^*_{s,T}(x))]\end{aligned}$$

where $x=\xi^*_{0,s}(x_0)$.

Similarly,

$$\begin{aligned}J(u)&=E[c(\xi^u_{0,T}(x_0))]\\&=E[c(\xi^u_{s,T}(x))]\\&=E[c(\xi^*_{s,T}(z_{s+h}))].\end{aligned}$$

Therefore,

$$J(u)-J(u^*)=E[c(\xi^*_{s,T}(z_{s+h}))-c(\xi^*_{s,T}(x))].$$

Because $\xi^*_{s,T}(\cdot)$ is differentiable this is

$$(7.1)\qquad =E\left[\int_s^{s+h} c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}}{\partial x}(z_r)\cdot\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}(f(r,\xi^*_{s,r}(z_r),u_r) - f(r,\xi^*_{s,r}(z_r),u^*_r))\,dr\right].$$

As in § 5, this gives an explicit formula for the change in the cost resulting from a "strong variation" in the optimal stochastic open loop control. It involves a time integration over $[s, s+h]$ and, again, the only remaining problem is to justify the differentiation of the right-hand side of (7.1).

Write

$$\Gamma(s,r,z_r)=c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}}{\partial x}(z_r)\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}$$

and

$$(7.2)\qquad \begin{aligned} p_s(x)&=E\left[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}}{\partial x}(x)\,|\,F_s\right]\\ &=E[\Gamma(s,s,x)\,|\,F_s],\end{aligned}$$

where, as above, $x=\xi^*_{0,s}(x_0)$.

Then arguments similar to those of § 5—but in fact simpler because Z is not involved—enable us to show that there is a set $S\subset[0,T]$ of zero Lebesque measure such that if $s\notin S$,

$$E[\Gamma(s,s,x)(f(s,\xi^*_{0,s}(x_0),u)-f(s,\xi^*_{0,s}(x_0),u^*))I_A]\geqq 0$$

for any $u\in V$ and $A\in F_s$.

That is, in terms of the adjoint variable $p_s(x)$ we have the following minimum principle for stochastic open loop controls.

THEOREM 7.1. *If $u^*\in V$ is an optimal stochastic open loop control there is a set $S\subset[0,T]$ of zero Lebesgue measure such that if $s\notin S$*

$$p_s(x)f(s,x,u^*)\leqq p_s(x)f(s,x,u)\quad a.s.$$

for all $u\in V$. That is, the optimal control u^* almost surely minimizes the Hamiltonian with adjoint variable $p_s(x)$.

Remark 7.2. Under certain conditions the minimum cost attainable under the stochastic open loop controls is equal to the minimum cost attainable under the Markov feedback controls of the form $u(s,\xi^u_{0,s}(x_0))$. See for example [3], [12]. If u_M is a Markov control, with a corresponding, possibly weak, solution trajectory ξ^{u_M}, then u_M can be considered as a stochastic open loop control $u_M(w)$ by putting

$$u_M(w)=u_M(s,\xi^{u_M}_{0,s}(x_0,w)).$$

This means the control in effect "follows" its original trajectory ξ^{u_M} rather than any new trajectory. That is, the control is similar to the adjoint strategies considered by Krylov [15]. The significance of this is that when we consider variations in the state trajectory ξ, and derivatives of the map $x\to\xi_{s,t}(x)$, the control does not react, and so we do not introduce derivatives in the u variable.

If the optimal control u^* is the Markov, then the process ξ^* is Markov and

$$p_s(x) = E[\Gamma(s,s,x)\,|\,F_s] \\ = E[\Gamma(s,s,x)\,|\,x]. \tag{7.3}$$

8. The adjoint process. Suppose the optimal stochastic open loop control u^* is Markov. The Jacobian $\partial\xi^*_{s,T}/\partial x$ exists, as does $(\partial\xi^*_{s,T}/\partial x)^{-1}$ and higher derivatives.

THEOREM 8.1. *Suppose the optimal control u^* is Markov. Then*

$$p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0))C_{0,t}] - \int_0^s p_r(\xi^*_{0,r}(x_0))f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)\,dr$$

$$+\int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r, \xi^*_{0,r}(x_0))\,dw_r$$

$$-\int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r, \xi^*_{0,r}(x_0))g_\xi(r, \xi^*_{0,r}(x_0))\,dr.$$

Proof. Write $f_\xi(r)$ for $f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)$ and $g(r)$ for $g(r, \xi^*_{0,r}(x_0))$, etc. By uniqueness of the solutions to (6.1)

$$\xi^*_{0,T}(x_0) = \xi^*_{s,T}(\xi^*_{0,s}(x_0)) \tag{8.1}$$

so, differentiating,

$$C_{0,T} = C_{s,T}C_{0,s} \tag{8.2}$$

where $C_{0,T} = C^*_{0,T}$, etc. (without the *).

From (7.2) and (7.3)

$$p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0))C_{s,T}\,|\,F_s],$$

so from (8.2)

$$p_s(x)C_{0,s} = E[c_\xi(\xi^*_{0,T}(x_0))C_{0,T}\,|\,F_s], \tag{8.3}$$

and this is a $(P,\{F_t\})$ martingale. Write $x = \xi^*_{0,s}(x_0)$, $C = C_{0,s}$. From the martingale representation result [10], the integrand in the representation of $p_s(x)C$ as a stochastic integral is obtained by the Itô rule, noting that only the stochastic integral terms will appear. These involve the derivatives in x and C. In fact, by considering the system $\bar\xi_{0,t}$ with components $\xi^*_{0,t}$ and $C_{0,t}$ and any real C^2 function Φ, the martingale

$$M_s = E[\Phi(\bar\xi_{0,T}\,|\,F_s] = E[\Phi(\bar\xi_{0,T})\,|\,x, C] = V(s,x,C)$$

$$= V(0, x_0, I) + \int_0^s V_x(r, \xi^*_{0,r}(x_0), C_{0,r})g(r)\,dw_r$$

$$+\sum_{k=1}^n \int_0^s V_C(r, \xi^*_{0,r}(x_0), C_{0,r})g^{(k)}_\xi(r)C_{0,r}\,dw^k_r.$$

Therefore, for the vector martingale (8.3)

$$p_s(x)C = E[c_\xi(\xi_{0,T}(x_0))C_{0,T}] + \int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r)\,dw_r C_{0,r} \\ +\sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g^{(k)}_\xi(r)C_{0,r}\,dw^k_r. \tag{8.4}$$

Recall that $D_{0,s}=C^{-1}$, so forming the product of (6.4) and (8.4) by using the Itô rule, we have

$$
\begin{aligned}
p_s(x) &= (p_s(x)C)D_{0,s} \\
&= E[c_\xi(\xi^*_{0,T}(x_0))C_{0,T}] - \int_0^s p_r(\xi^*_{0,r}(x_0))f_\xi(r)\,dr \\
&\quad - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)\,dw_r^k + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2\,dr \\
&\quad + \int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r)\,dw_r + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)\,dw_r^k \\
&\quad - \sum_{k=1}^n \int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)\,dr - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2\,dr \\
&= E[c_\xi(\xi^*_{0,T}(x_0))C_{0,T}] - \int_0^s p_r(\xi^*_{0,r}(x_0))f_\xi(r)\,dr \\
&\quad + \int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r)\,dw_r - \sum_{k=1}^n \int_0^s p_x(r, \xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)\,dr,
\end{aligned}
$$

thus establishing the result.

This verifies by a simple, direct method the formula of Haussmann [12] without any requirement that the diffusion coefficient matrix gg^* is nonsingular. However we do not identify $p_s(x)$ with the gradient of the minimum cost process; this follows from arguments as in [12].

9. **Conclusion.** Using the theory of stochastic flows the effect of a perturbation of an optimal control is explicitly calculated in both the partially observed and stochastic open loop cases. The only difficulty is to justify the differentiation. The adjoint variable $p_s(x)$ is explicitly identified.

THEOREM 9.1. *If f is differentiable in the control variable u, and if the random variable $x=\xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under the measure P^*, then the inequality of Theorem 5.1 implies*

$$\sum_{j=1}^k (u_j(x)-u_j^*(s)) \int_{R^d} \Gamma(s,x)\frac{\partial f}{\partial u_j}(s,x,u^*)q_s(x)\,dx \geqq 0.$$

This is the result of Bensoussan's paper [1].

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The Adjoint Process in Stochastic Optimal Control

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Abstract. Using stochastic flows a minimum principle is obtained when a diffusion is controlled using stochastic open loop controls. An equation for the adjoint process is then derived using an explicit formula for the integrand in a certain stochastic integral.

1. Introduction.

There have been many proofs of minimum principles in stochastic control. For a small example see the works of Kushner [15], Bismut [2], Haussmann [10], [11], [12], Davis and Varaiya [6], and the book by Elliott [8]. In this paper we consider a diffusion and stochastic open loop controls, that is, controls which are adapted to the filtration of the driving Brownian motion process. For such controls the dynamical equations have strong solutions, and the results on the differentiability of the solution, due originally to Blagoveschenskii and Freidlin [1], can be applied. The work of Kunita [14] and Bismut [2] on stochastic flows enables the variation in the expected cost, due to a perturbation of the optimal control, to be calculated explicitly. The minimum principle follows by differentiating this quantity.

If the optimal control is Markov the stochastic integral representation result of [9] is applied to give an expression for a quantity associated with the adjoint process. Stochastic calculus is then used to derive the equation satisfied by the adjoint process.

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2. Dynamics.

Suppose the state of a system is described by a stochastic differential equation:

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t$$
$$\xi_t \in R^d, \quad \xi_0 = x_0, \quad 0 \leq t \leq T. \tag{2.1}$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. We shall make the following assumptions.

A_1: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x and for some constant K

$$(1 + |x|)^{-1}|f(t,x,u)| + |f_x(t,x,u)| \leq K_1.$$

A_2: $g : [0,T] \times R^d \to R^d \otimes R^n$ is a matrix valued, Borel measurable function, continuously differentiable in x, and for some constant K_2

$$|g(t,x)| + |g_x(t,x)| \leq K_2.$$

The columns of g will be denoted by $g^{(k)}$ for $k = 1, \ldots, n$.

A_3: $w = (w^1, \ldots, w^n)$ is an n-dimensional Brownian motion on a probability space (Ω, F, P) with a right continuous, complete filtration $\{F_t\}$, $0 \leq t \leq T$.

DEFINITION 2.1. *The set of admissible controls $\underline{U}$ will be the F_t-predictable functions on $[0,T] \times \Omega$ with values in U. These are sometimes called 'stochastic open loop' controls, [3].*

REMARKS 2.2. For each $u \in \underline{U}$ there is, therefore, a strong solution of (2.1), and we shall write $\xi^u_{s,t}(x)$ for the solution trajectory given by

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)dr + \int_s^t g(r, \xi^u_{s,r}(x))dw_r. \tag{2.2}$$

Then, because u is a (predictable) parameter, the result of Blagovenscenskii and Freidlin [1] extends to this situation, so the Jacobian $\frac{\partial \xi^u_{s,t}}{\partial x}(x) = D^u_{s,t}$ exists and is the solution of

$$D^u_{s,t} = I + \int_s^t f_\xi(r, \xi^u_{s,r}(x), u_r)D^u_{s,r}\,dr + \sum_{k=1}^n \int_s^t g^{(k)}_\xi(r, \xi^u_{s,r}(x))D^u_{s,r}\,dw^k_r. \tag{2.3}$$

Here I is the $d \times d$ identity matrix. In fact, if the coefficients f and g are C^k the map $x \to \xi^u_{s,t}(x)$ is C^{k-1}.

Consider the matrix valued process H defined by:

$$H^u_{s,t} = I - \int_s^t H^u_{s,r}\big(f_\xi(r, \xi^u_{s,r}(x), u_r) - \sum_{k=1}^n g^{(k)}_\xi(r, \xi^u_{s,r}(x))^2\big)dr$$
$$- \sum_{k=1}^n \int_s^t H^u_{s,r} g^{(k)}_\xi(r, \xi^u_{s,r}(x))dw^k_r. \tag{2.4}$$

Then using the Ito rule we see $d(H^u_{s,t} D^u_{s,t}) = 0$ and $H^u_{ss} D^u_{ss} = I$, so $H^u_{s,t} = (D^u_{s,t})^{-1}$.

Write $\|\xi^u(x_0)\|_t = \sup_{0 \le s \le t} |\xi^u_{0,s}(x_0)|$. Then, as in Lemma 2.1 of [12], for any p, $1 \le p < \infty$, using Gronwall's and Jensen's inequalities

$$\|\xi^u(x_0)\|^p_T \le C\Big(1 + |x_0|^p + \Big|\int_0^t g(r, \xi^u_{0,r}(x_0))dw_r\Big|^p\Big)$$

almost surely for some constant C. Therefore, using Burkholder's inequality and hypothesis A_2, $\|\xi^u(x_0)\|_T$ is in L^p for all p, $1 \le p < \infty$. Write

$$\|D^u\|_T = \sup_{0 \le s \le T} |D^u_{0,s}|$$
$$\|H^u\|_T = \sup_{0 \le s \le T} |H^u_{0,s}|.$$

Then, because f_ξ and g_ξ are bounded, an application of Gronwall's, Jensen's and Burkholder's inequalities again implies

$$\|D^u\|_T \text{ and } \|H^u\|_T \quad \text{are in } L^p \text{ for all } p, \quad 1 \le p < \infty.$$

COST 2.3. Suppose for simplicity that the cost associated with the process is purely terminal and given by a bounded C^2 function

$$c(\xi^u_{0,T}(x_0)).$$

A_4: We suppose $|c(x)| + |c_x(x)| + |c_{xx}(x)| \le K_3(1 + |x|^q)$ for some $q < \infty$.

The expected cost if a control $u \in \underline{U}$ is used is, therefore,

$$J(u) = E[c(\xi^u_{0,T}(x_0))].$$

We shall suppose there is an optimal control $u^* \in \underline{U}$ so

$$J(u^*) \leq J(u) \quad \text{for all} \quad u \in \underline{U}.$$

NOTATION 2.4. *If u^* is an optimal control write ξ^* for ξ^{u^*}, D^* for D^{u^*} etc.*

REMARKS 2.5. Consider a d-dimensional semimartingale of the form

$$z_t = z_s + A_t$$

where A is a predictable bounded variation process. Then Kunita's formula [14] for the composition of processes can be applied, (see also Bismut [5]), and we have

$$\begin{aligned} \xi^*_{s,t}(z_t) = z_s &+ \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr \\ &+ \int_s^t \frac{\partial \xi^*_{s,r}}{\partial x}(z_r)dA_r + \sum_{k=1}^n \int_s^t g^{(k)}(r, \xi^*_{s,r}(z_r))dw^k_r. \end{aligned} \tag{2.5}$$

DEFINITION 2.6. *Consider perturbations of the optimal control u^* of the following kind: For $s \in [0,T]$, $h > 0$ such that $0 \leq s < s + h < T$, and $A \in F_s$ define, for any other admissible control $\tilde{u} \in \underline{U}$,*

$$u(t,w) = \begin{cases} u^*(t,w) & \text{if } (t,w) \notin [s, s+h] \times A \\ \tilde{u}(t,w) & \text{if } (t,w) \in [s, s+h] \times A. \end{cases}$$

Applying (2.5) we have, similarly to Theorem 5.1 of [4], the following result.

THEOREM 2.7. *For the perturbation u of u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr. \tag{2.6}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

PROOF. Substituting (2.6) in (2.5) we see

$$
\begin{aligned}
\xi^*_{s,t}(z_t) = x &+ \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr \\
&+ \int_s^t \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\right) \left(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\right)^{-1} \Big(f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r)\Big)dr \\
&+ \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r \\
= x &+ \int_s^t f(r, \xi^*_{s,r}(z_r), u_r)dr + \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r.
\end{aligned}
$$

However, the solution to (2.2) is unique so $\xi^*_{s,t}(z_r) = \xi^u_{s,t}(x)$.

REMARKS 2.8. Note that $u(t) = u^*(t)$ if $t > s+h$ so $z_t = z_{s+h}$ if $t > s+h$. Therefore

$$
\xi^*_{s,t}(z_t) = \xi^*_{s,t}(z_{s+h}) = \xi^*_{s+h,t}(\xi^u_{s,s+h}(x))
$$

if $t > s+h$.

3. A Minimum Principle.

Now

$$
\begin{aligned}
J(u^*) &= E[c(\xi^*_{0,T}(x_0))] \\
&= E[c(\xi^*_{s,T}(x))] \quad \text{where } x = \xi_{0,s}(x_0),
\end{aligned}
$$

because, by uniqueness, $\xi^*_{0,T}(x_0) = \xi^*_{s,T}(x)$. Similarly,

$$
\begin{aligned}
J(u) &= E[c(\xi^u_{0,T}(x_0))] \\
&= E[c(\xi^u_{s,T}(x))] \\
&= E[c(\xi^*_{s,T}(z_{s+h}))].
\end{aligned}
$$

Therefore,

$$
J(u) - J(u^*) = E[c(\xi^*_{s,T}(z_{s+h})) - c(\xi^*_{s,T}(x))].
$$

Because $\xi^*_{s,T}(\cdot)$ is differentiable this is

$$
= E\Big[\int_s^{s+h} c_\xi(\xi^*_{s,T}(z_r))\frac{\partial \xi^*_{s,T}(z_r)}{\partial x}\cdot\Big(\frac{\partial \xi^*_{s,r}(z_r)}{\partial x}\Big)^{-1}\ (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr\Big]. \tag{3.1}
$$

This gives an explicit formula for the change in the cost resulting from a 'strong' variation in the optimal control. It involves only a time integration. The only remaining problem is to justify the differentiation of the right hand side of (3.1).

Write $\Gamma(s,r,z_r) = c_\xi(\xi^*_{s,T}(z_r))\dfrac{\partial \xi^*_{s,T}(z_r)}{\partial x}\Big(\dfrac{\partial \xi^*_{s,r}(z_r)}{\partial x}\Big)^{-1}$.

Then

$$
\begin{aligned}
J(u) - J(u^*) = &\int_s^{s+h} E\Big[\big(\Gamma(s,r,z_r) - \Gamma(s,r,x)\big)\big(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r)\big)\Big]dr \\
&+ \int_s^{s+h} E\Big[\big(\Gamma(s,r,x) - \Gamma(r,r,x)\big)\big(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r)\big)\Big]dr \\
&+ \int_s^{s+h} E\Big[\Gamma(r,r,x)(f(r,\ \xi^*_{s,r}(z_r),\ u_r) - f(r,\ \xi^*_{s,r}(z_r),\ u^*_r) \\
&\qquad - f(r,\ \xi^*_{s,r}(x),\ u_r) + f(r,\ \xi^*_{s,r}(x),\ u^*_r))\Big]dr \\
&+ \int_s^{s+h} E\Big[\Gamma(r,r,x)(f(r,\ \xi^*_{0,r}(x_0),\ u_r) - f(r, \xi^*_{0,r}(x_0),\ u^*_r))\Big]dr \\
= &\ I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.}
\end{aligned}
$$

Now,

$$
\begin{aligned}
|I_1(h)| &\le K_4 \int_s^{s+h} E\Big[|\Gamma(s,r,z_r) - \Gamma(s,r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]\,dr \\
&\le K_4 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,r,z_r) - \Gamma(s,r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big] \\
|I_2(h)| &\le K_5 \int_s^{s+h} E\Big[|\Gamma(s,r,x) - \Gamma(r,r,x)|(1 + \|\xi^u(x_0)\|_{s+h})\Big]\,dr \\
&\le K_5 h \sup_{s \le r \le s+h} E\Big[|\Gamma(s,r,z_r) - \Gamma(r,r,x)|(1 + \|\xi^u(x_0)\|_{s+h}\Big] \\
|I_3(h)| &\le K_6 \int_s^{s+h} E\Big[|\Gamma(r,r,x)|\ \|x - z_r\|\Big]\,dr \\
&\le K_6 h \sup_{s \le r \le s+h} E\Big[|\Gamma(r,r,x)|\ \|x - z.\|_{s+h}\Big].
\end{aligned}
$$

The differences $|\Gamma(s,r,z_r) - \Gamma(s,r,x)|$, $|\Gamma(s,r,x) - \Gamma(r,r,x)|$ and $\|x - z\|_{s+h}$ are all uniformly bounded in some L^p, $p > 1$, and

$$
\begin{aligned}
&\lim_{r \to s} |\Gamma(s,r,z_r) - \Gamma(s,r,x)| = 0 \quad \text{a.s.} \\
&\lim_{r \to s} |\Gamma(s,r,x) - \Gamma(r,r,x)| = 0 \quad \text{a.s.} \\
&\lim_{h \to 0} \|x - z.\|_{s+h} = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{r \to s} \|\Gamma(s,r,z_r) - \Gamma(s,r,x)\|_p = 0 \\
&\lim_{r \to s} \|\Gamma(s,r,x) - \Gamma(r,r,x)\|_p = 0 \\
\text{and} \quad &\lim_{h \to 0} \|(\|x - z\|_{s+h})\|_p = 0 \quad \text{for some } p.
\end{aligned}
$$

Consequently, $\lim_{h \to 0} h^{-1} I_k(h) = 0$, for $k = 1, 2, 3$.

The only remaining problem concerns the differentiability of

$$
I_4(h) = \int_s^{s+h} E\Big[\Gamma(r,r,x)(f(r,\ \xi^*_{0,r}(x_0),\ u_r) - f(r,\ \xi^*_{0,r}(x_0),\ u^*_r))\Big]\,dr.
$$

The integrand is almost surely in $L^1([0,T])$ so $\lim_{h \to 0} h^{-1} I_4(h)$ exists for almost every $s \in [0,T]$. However, the set of times $\{s\}$ where the limit may not exist might depend on the

control u. Consequently we must restrict the perturbations u of the optimal control u^* to perturbations from a countable dense set of controls. In fact:

1) Because the trajectories are, almost surely, continuous, F_ρ is countably generated by sets $\{A_{i\rho}\}$, $i = 1, 2, \ldots$ for any rational number $\rho \in [0, T]$. Consequently F_t is countably generated by the sets $\{A_{i\rho}\}$, $r \leq t$.

2) Let G_t denote the set of measurable functions from (Ω, F_t) to $U \subset R^k$. (If $u \in \underline{U}$ then $u(t, w) \in G_t$.) Using the L^1-norm, as in [7], there is a countable dense subset $H_\rho = \{u_{j\rho}\}$ of G_ρ, for rational $\rho \in [0, T]$. If $H_t = \bigcup_{\rho \leq t} H_\rho$ then H_t is a countable dense subset of G_t. If $u_{j\rho} \in H_\rho$ then, as a function constant in time, $u_{j\rho}$ can be considered as an admissible control over any time interval $[t, T]$ for $t \geq \rho$.

3) The countable family of perturbations is obtained by considering sets $A_{i\rho} \in F_t$, functions $u_{j\rho} \in H_t$, where $\rho \leq t$, and defining as in 3.1

$$u^*_{j\rho}(s, w) = \begin{cases} u^*(s, w) & \text{if } (s, w) \notin [t, T] \times A_{i\rho} \\ u_{j\rho}(s, w) & \text{if } (s, w) \in [t, T] \times A_{i\rho}. \end{cases}$$

Then for each i, j, ρ

$$\lim_{h \to 0} h^{-1} \int_s^{s+h} E\Big[\Gamma(r, r, x)(f(r,\ \xi^*_{0,r}(x_0),\ u^*_{j\rho}) - f(r,\ \xi^*_{0,r}(x_0),\ u^*))\Big] dr \tag{3.2}$$

exists and equals

$$E\Big[\Gamma(s, s, x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*)) I_{A_{i\rho}}\Big]$$

for almost all $s \in [0, T]$.

Therefore, considering this perturbation we have

$$\lim_{h \to 0} h^{-1} (J(u^*_{j\rho}) - J(u^*)) = E\Big[\Gamma(s, s, x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*)) I_{A_{i\rho}}\Big]$$
$$\geq 0 \quad \text{for almost all } s \in [0, T].$$

Consequently there is a set $S \subset [0, T]$ of zero Lebesgue measure such that, if $s \notin S$, the limit in (3.2) exists for all i, j, ρ, and gives

$$E\Big[\Gamma(s, s, x)(f(s,\ \xi^*_{0,s}(x_0),\ u_{j\rho}) - f(s,\ \xi^*_{0,s}(x_0),\ u^*)) I_{A_{i\rho}}\Big] \geq 0.$$

Using the monotone class theorem, and approximating an arbitrary admissible control $u \in \underline{U}$ we can deduce that if $s \notin S$

$$E\Big[\Gamma(s,s,x)\big(f(s,\ \xi^*_{0,s}(x_0),\ u) - f(s,\ \xi^*_{0,s}(x_0),\ u^*)\big)I_A\Big] \geq 0 \quad \text{for any } u \in U \text{ and } A \in F_s. \tag{3.3}$$

Write

$$p_s(x) = E\Big[c_\xi(\xi^*_{0,T}(x_0))\frac{\partial \xi^*_{s,T}(x)}{\partial x} \mid F_s\Big] = E[\Gamma(s,s,x) \mid F_s] \tag{3.4}$$

where, as before, $x = \xi^*_{0,s}(x_0)$. Then $p_t(x)$ is the adjoint variable and we have in (3.3) proved the following minimum principle:

THEOREM 5.1. *If $u^* \in \underline{U}$ is an optimal control there is a set $S \subset [0,T]$ of zero Lebesgue measure such that if $s \notin S$*

$$p_s(x)f(s,x,u^*) \leq p_s(x)f(s,x,u) \quad \text{a.s.}$$

That is, the optimal control u^* almost surely minimizes the Hamiltonian and the adjoint variable is $p_s(x)$.

REMARKS 3.2. Under certain conditions the minimum cost attainable under the stochastic open loop controls is equal to the minimum cost attainable under the Markov, feedback controls of the form $u(s, \xi^u_{0,s}(x_0))$. See for example [2], [10]. If u_M is a Markov control, with a corresponding, possibly weak, solution trajectory ξ^{u_M}, then u_M can be considered as a stochastic open loop control $u_M(w)$ by putting

$$u_M(w) = u_M(s, \xi^{u_M}_{0,s}(x_0, w)).$$

This means the control in effect 'follows' its original trajectory ξ^{u_M} than any new trajectory. That is the control is similar to the adjoint strategies considered by Krylov [13]. The significance of this is that when we consider variations in the state trajectory ξ, and derivatives of the map $x \to \xi_{s,t}(x)$, the control does not react, and so we do not introduce derivatives in the u variable.

If the optimal control u^* is Markov the process ξ^* is Markov and

$$\begin{aligned} p_s(x) &= E[\Gamma(s,s,x) \mid F_s] \\ &= E[\Gamma(s,s,x) \mid x]. \end{aligned} \tag{3.5}$$

4. The Adjoint Process.

Suppose the optimal control u^* is Markov. As noted above, u^* can and will be considered as an open loop control. The Jacobian $\frac{\partial \xi^*_{s,T}}{\partial x}$ exists, as does $\left(\frac{\partial \xi^*_{s,T}}{\partial x}\right)^{-1}$ and higher derivatives.

THEOREM 4.1. *Suppose the optimal control u^* is Markov. Then*

$$\begin{aligned} p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0))D_{0,T}] &- \int_0^s p_r(\xi^*_{0,r}(x_0)) f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)dr \\ &+ \int_0^s p_x(r, \xi^*_{0,r}(x_0)) g(r, \xi^*_{0,r}(x_0)) dw_r \\ &- \int_0^s p_x(r, \xi^*_{0,r}(x_0)) g(r, \xi^*_{0,r}(x_0)) g_\xi(r, \xi^*_{0,r}(x_0)) dr. \end{aligned}$$

PROOF. Write $f_\xi(r)$ for $f_\xi(r, \xi^*_{0,r}(x_0), u^*_r)$ and $g(r)$ for $g(r, \xi^*_{0,r}(x_0))$, etc. By uniqueness of the solutions to (2.1)

$$\xi^*_{0,T}(x_0) = \xi^*_{s,T}(\xi^*_{0,s}(x_0)) \tag{4.1}$$

so, differentiating,

$$D_{0,T} = D_{s,T} D_{0,s} \tag{4.2}$$

where $D_{0,T} = D^*_{0,T}$ etc. (without the *).

From (3.4) and (3.5)

$$p_s(x) = E[c_\xi(\xi^*_{0,T}(x_0))D_{s,T} \mid F_s]$$

so from (4.2)

$$p_s(x) D_{0,s} = E[c_\xi(\xi^*_{0,T}(x_0))D_{0,T} \mid F_s] \tag{4.3}$$

and this is a $(P,\{F_t\})$ martingale. Write $x = \xi^*_{0,s}(x_0)$, $D = D_{0,s}$. From the martingale representation result [9], the integrand in the representation of $p_s(x)D$ as a stochastic integral is obtained by the Ito rule, noting that only the stochastic integral terms will appear. These involve the derivatives in x and D. Therefore

$$\begin{aligned} p_s(x)D &= E[c_\xi(\xi_{0,T}(x_0))D_{0,T}] + \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)dw_r D_{0,r} \\ &\quad + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)D_{0,r}\,dw_r^k. \end{aligned} \tag{4.4}$$

Recall from (2.4) that $H_{0,s} = D^{-1}$ so forming the product of (2.4) and (4.4), using the Ito rule:

$$\begin{aligned} p_s(x) &= (p_s(x)D)H_{0,s} \\ &= E[c_\xi(\xi^*_{0,T}(x_0))D_{0,T}] - \int_0^s p_r(\xi^*_{0,r}(x_0))f_\xi(r)dr \\ &\quad - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)dw_r^k + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2 dr \\ &\quad + \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)dw_r + \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))g_\xi^{(k)}(r)dw_r^k \\ &\quad - \sum_{k=1}^n \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)dr - \sum_{k=1}^n \int_0^s p_r(\xi^*_{0,r}(x_0))(g_\xi^{(k)}(r))^2 dr \\ &= E[c_\xi(\xi^*_{0,T}(x_0))D_{0,T}] - \int_0^s p_r(\xi^*_{0,T}(x_0))f_\xi(r)dr \\ &\quad + \int_0^2 p_x(r,\xi^*_{0,r}(x_0))g(r)dw_r - \sum_{k=1}^n \int_0^s p_x(r,\xi^*_{0,r}(x_0))g(r)g_\xi^{(k)}(r)dr \end{aligned}$$

so establishing the result.

This verifies by a simple, direct method the formula of Haussman [10].

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Direct Solutions of Kolmogorov's Equations by Stochastic Flows

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Solutions of Kolmogorov's forward and backward equations are obtained by considering a family of conditional expectations and the use of stochastic flows to justify differentiation in the time variable. © 1989 Academic Press, Inc.

INTRODUCTION

Probabilistic solutions of the Cauchy problem for Kolmogorov's forward and backward equations have been known for many years. In [5, 6] Kunita uses stochastic flows associated with forward and backward stochastic differential equations to write down explicit forms of the solutions. In fact he uses both forms simultaneously, in that he requires the backward equation for the forward process to solve the Kolmogorov backward equation, and the forward equation for the backward process to solve the Kolmogorov forward equation. In this note we indicate how solutions of the Kolmogorov equations can be obtained directly by differentiation in the time variable of a family of conditional expectations. This is justified by differentiating inside the conditional expectation, using the properties of stochastic flows. Both the forward and backward Kolmogorov equations are considered. As noted in [3], the conditional expectation is a solution of Kolmogorov's equation because the bounded variation term in its semimartingale representation, using the Ito formula, is zero.

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1. Stochastic Flows

Let $w_t=(w_t^1, ..., w_t^m)$ be m-dimensional standard Brownian motion defined for all $t\geqslant 0$ on a fixed probability space (Ω, F, P). Write $(F_t)_{t\geqslant 0}$ for the right-continuous complete filtration generated by w.

Suppose that we are given vector fields $X_0, X_1, ..., X_m$ on $[0, \infty)\times R^d$ such that each X_i is three times continuously differentiable with all derivatives bounded. We can associate a stochastic flow with w and the X_i:

Theorem 1.1 [2]. *There is a map* $\xi: [0, \infty)\times[0, \infty)\times\mathbb{R}^d\times\Omega\to{}^d\mathbb{R}$ *such that*

(i) *for* $0\leqslant s\leqslant t$ *and* $x\in\mathbb{R}^d$, $\xi_{st}(x)$ *is the essentially unique solution of the stochastic differential equation*

$$d\xi_{s,t}(x)=X_0(t, \xi_{s,t}(x))\,dt+X_i(t, \xi_{s,t}(x))\,dw_t^i \tag{1.1}$$

with $\xi_{s,s}(x)=x$ *(we employ the Einstein summation convention when convenient);*

(ii) *for each s and t and almost all w, the map* $\xi_{s,t}(\cdot)$ *has a version which is twice differentiable in x and continuous in s and t. The Jacobian* $D_{s,t}=\partial\xi_{s,t}/\partial x$ *is then a solution of the linearized equation*

$$dD_{s,t}=\frac{\partial X_0}{\partial\xi}(t, \xi_{s,t}(x))\,D_{s,t}\,dt+\frac{\partial X_i}{\partial\xi}(t, \xi_{s,t}(x))\,D_{s,t}\,dw_t^i; \tag{1.2}$$

(iii) *the second derivative* $\partial^2\xi_{s,t}/\partial x^2$ *is well-defined for each x, s, t.*

Remark. The conditions we have imposed on the coefficients of (1.1) suffice to ensure that $\xi_{s,t}$ and $D_{s,t}$ belong to $L^p(\Omega)$ for all $p>0$. The arguments to establish appropriate estimates are standard applications of the Burkholder, Jensen, and Gronwall inequalities, similar to those in [1, 4]. The conditions on the coefficients can be relaxed somewhat: those employed here are fixed in the interests of simplicity.

Now restrict attention for the remainder of this paper to the time interval $[0, T]$, where T is a fixed finite positive number. Consider the trajectories with a fixed initial position $x_0\in\mathbb{R}^d$, and assume that we are given a function $c(\xi_{0,T}(x_0))$ of the final position at $t=T$. Assume further that c is three times continuously differentiable with bounded derivatives.

Define the (F_t)-martingale $M=(M_t)_{0\leqslant t\leqslant T}$ by setting

$$M_t=E[c(\xi_{0,T}(x_0))\,|\,F_t].$$

It is shown in [3] that the uniqueness of the martingale representation

$M_t = M_0 + \int_0^t \gamma_i(s)\, dW_s^i$, where the integrands $\gamma_i(s)$ are predictable processes, can be exploited to find an explicit expression for $\gamma_i(s)$ and to solve the Cauchy problem: in fact, if we set $x = \xi_{0,t}(x_0)$ and write (as in Section 2 below) $M_t = E_{t,x}[c(\xi_{t,T}(x))] = V(t, x)$, then it is shown that V is a solution of the backward parabolic differential equation $\partial V/\partial t + LV = 0$, with final condition $V(T, u_T) = c(x_T)$, where L is the operator

$$L = \sum_{i=1}^{d} X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\sum_{k=1}^{m} (X_k^i X_k^j) \left(\frac{\partial^2}{\partial x_i\, \partial x_j} \right) \right).$$

In [3] the Ito differention rule is applied to $V(t,)x)$ to yield these results. This requires the differentiability of V in t, which can be proved using Kunita's results [6] on the reverse time stochastic differential equation for the reverse flow $\hat{\xi}_{s,t}(\cdot)$. Our purpose in this note is to show that this can be avoided by a direct calculation of $\partial V/\partial t$. This approach has the advantage that only the semigroup properties of the flow $\xi_{s,t}$, the Markov property, and the independence of Brownian increments are required for the proof.

2. Differentiability

Because $w_0 = 0$ for standard m-dimensional Brownian motion we observe that

$$\begin{aligned} F_t &= \sigma\{w_s : 0 \leqslant s \leqslant t\} \\ &= F_t^0 = \sigma\{w_s - w_r : 0 \leqslant r \leqslant s \leqslant t\}. \end{aligned}$$

More generally we shall write $F_t^u = \sigma\{w_s - w_r : u \leqslant s \leqslant t\}$. Therefore, if the initial condition x_0 of the trajectory is known, by the Markov property and independence of future increments

$$\begin{aligned} E[c(\xi_{0,T}(x_0)) \mid F_t] &= E[c(\xi_{0,T}(x_0)) \mid F_t^0] \\ &= E[c(\xi_{0,T}(x_0)) \mid x] \\ &= E_{t,x}[c(\xi_{0,T}(x_0))] \\ &= V(t, x), \qquad \text{where} \quad x = \xi_{0,t}(x_0). \end{aligned}$$

Now by the uniqueness of solutions of (1.1), and the semigroup property,

$$\begin{aligned} \xi_{0,T}(x_0) &= \xi_{t,T}(\xi_{0,t}(x_0)) \\ &= \xi_{t,T}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} V(t, x) &= E_{t,x}[c(\xi_{t,T}(x))] \\ &= E[c(\xi_{t,T}(x)) \mid F_t]. \end{aligned} \tag{2.1}$$

Write

$$D_{s,t} = \frac{\partial \xi_{s,t}}{\partial x_s}(x_s), \qquad W_{s,t} = \frac{\partial^2 \xi_{s,t}}{\partial x_2^2}(x_s).$$

Differentiating (2.1) in x we see

$$\frac{\partial V}{\partial x}(t, x) = V_x(t, x) = E[c_\xi(\xi_{t,T}(x))\, D_{t,T} \mid F_t]$$

$$\frac{\partial^2 V}{\partial x^2}(t, x) = V_{xx}(t, x) = E[c_{\xi\xi}(\xi_{t,T}(x))\, D_{t,T} \otimes D_{t,T}]$$

$$+ [c_{\xi\xi}(\xi_{t,T}(x))\, W_{t,T} \mid F_t],$$

because the map $x \to \xi_{t,T}(x)$ is C^2.

We now wish to investigate the differentiability of $V(t, x)$ in the t variable. Under strong enough conditions on the coefficients X_0, X_i this would follow as in [6] by considering the reverse time stochastic differential equation for the reverse flow $\xi_{s,t}(\cdot)$. However, we will give a direct proof.

THEOREM 2.1. *For any $x \in R^d$, $V(t, x)$ is continuously differentiable in t and*

$$\begin{aligned} -\frac{\partial V}{\partial x}(t, x) &= \sum_{i=1}^{d} X_0^i(t, x)\, E\left[c_\xi(\xi_{t,T}(x)) \frac{\partial \xi_{t,T}}{\partial x^i} \right] \\ &\quad + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} X_j^k(t, x)\, X_j^l(t, x)\, E[\partial_k\, \partial_l(c(\xi_{t,T}(x))] \\ &= LV(t, x), \qquad \textit{where} \quad \partial_k = \partial/\partial x_k. \end{aligned}$$

Proof. For a given $x \in R^d$ consider

$$V(t, x) = E[c(\xi_{t,T}(x))].$$

Using the mean value theorem, for $h>0$,

$$\begin{aligned}
c(\xi_{t-h,T}(x)) &- c(\xi_{t,T}(x)) \\
&= c(\xi_{t,T}(\xi_{t-h,t}(x))) - c(\xi_{t,T}(x)) \\
&= c_\xi(\xi_{t,T}(x))\frac{\partial \xi_{t,T}}{\partial x}\cdot(\xi_{t-h,t}(x)-x) \\
&\quad + \frac{1}{2}\sum_{k,l=1}^{d} \partial_k\,\partial_l(c(\xi_{t,T}(\cdot)))(\eta)(\xi^l_{t-h,t}(x)-x^l)(\xi^m_{t-h,t}(x)-x^m),
\end{aligned}$$

where $|\eta - x| \leqslant |\xi_{t-h,t}(x)-x)|$ and $x=(x^1,\ldots,x^d)$, where $x\in R^d$.

Now

$$\begin{aligned}
\xi_{t-h,t}(x) = x &+ \int_{t-h}^{t} X_0(r, \xi_{t-h,r}(x))\,dr \\
&+ \int_{t-h}^{t} X_i(r, \xi_{t-h,r}(x))\,dw^i_r,
\end{aligned}$$

so $(\xi_{t-h,t}(x)-x)$ is F^{t-h}_t measurable. For the given x, $\xi_{t,T}(x)$ and $\partial\xi_{t,T}/\partial x$ are F^t_T measurable, so because the increments of Brownian motion are independent

$$\begin{aligned}
E&\left[c_\xi(\xi_{t,T}(x_0))\frac{\partial \xi_{t,T}}{\partial x}\cdot(\xi_{t-h,t}(x)-x)\right] \\
&= E\left[c_\xi(\xi_{t,T}(\xi_{t,T}(x)))\frac{\partial \xi_{t,T}}{\partial x}\right]E\left[\int_{t-h}^{t} X_0(r,\xi_{t-h,r}(x))\,dr\right] \\
&= o(h). \qquad (2.2)
\end{aligned}$$

Write

$$\begin{aligned}
\sum_{k,l=1}^{d} &\partial_k\,\partial_l(c(\xi_{t,T}(\cdot)))(\eta)(\xi^l_{t-h,t}(x)-x^l)(\xi^m_{t-h,t}(x)-x^m) \\
&= \sum_{k,l=1}^{d} (\partial_k\,\partial_l(c(\xi_{t,T}(\cdot)))(\eta) \\
&\quad - \partial_k\,\partial_l(c(\xi_{t,T}(\cdot)))(x))(\xi^l_{t-h,t}(x)-x^l)(\xi^m_{t-h,t}(x)-x^m) \\
&\quad + \sum_{k,l=1}^{d} \partial_k\partial_l(c(\xi_{t,T}(\cdot)))(x)(\xi^l_{t-h,t}(x)-x^l) \\
&\quad \times(\xi^m_{t-h,t}(x)-x^m) = S_1+S_2, \qquad \text{say.}
\end{aligned}$$

Because the second derivatives of c are bounded, and because ξ and D belong to $L^p(\Omega)$ for all p,

$$\begin{aligned}
E[S_1] &\leqslant K_1\left(E\left[\left(\frac{\partial^2}{\partial x^2}(c\circ\xi_{t,T})(\eta)-\frac{\partial^2}{\partial x^2}(c\circ\xi_{t,T})(x)\right)^2\right]\right)^{1/2}\\
&\quad\cdot(E[|\xi_{t-h,t}(x)-x|^4]^{1/2})\\
&\leqslant K_2\left(E\left[\left(\frac{\partial^2}{\partial x^2}(c\circ\xi_{t,T})(\eta)-\frac{\partial^2}{\partial x^2}(c\circ\xi_{t,T})(x)\right)^2\right]\right)^{1/2} h\\
&= o(h) \qquad (2.3)
\end{aligned}$$

because, as $h\to 0$, $(\partial^2/\partial x^2)(c\circ\xi_{t,T})(\eta)-(\partial^2/\partial x^2)(c\circ\xi_{t,T})(x)$, converges to zero a.s. and is bounded. Again, because F_t^{t-h} and F_T^t are independent

$$\begin{aligned}
E[S_2] &= \sum_{k,l=1}^{d} E[\partial_k\,\partial_l(c\circ\xi_{t,T})(x)]\,E[(\xi^l_{t-h,t}(x)-x^l)(\xi^m_{t-h,t}(x)-x^m)]\\
&= \sum_{k,l=1}^{d} E[\partial_k\,\partial_l(c\circ\xi_{t,T})(x)]\\
&\quad\times\int_{t-h}^{t}\sum_{j=1}^{m} E[X_j^k(r,\xi_{t-h,r}(x))\,X_j^l(r,\xi_{t-h,r})]\,dr\\
&= o(h). \qquad (2.4)
\end{aligned}$$

From (2.2), (2.3), and (2.4) dividing by $h>0$ we have

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}(V(t-h,x)-V(t,x))\\
&\quad= X_0(t,x)\,E\left[c_\xi(\xi_{t,T}(x))\frac{\partial\xi_{t,T}}{\partial x}\right]\\
&\qquad+\frac{1}{2}\sum_{k,l=1}^{d}\sum_{j=1}^{m} X_j^k(t,x)\,X_j^l(t,x)\,E[\partial_k\,\partial_l(c\circ\xi_{t,T})(x)].
\end{aligned}$$

This establishes the existence of the left-hand derivative of V at t. The existence of the right-hand derivative is proved by considering

$$\begin{aligned}
&h^{-1}(V(t,x)-V(t+h,x))\\
&\quad= h^{-1}(E[c(\xi_{t,T}(x))]-E[c(\xi_{t+h,T}(x))])\\
&\quad= h^{-1}(E[c(\xi_{t+h,T}(\xi_{t,t+h}(x)))]-E[c(\xi_{t+h,T}(x))]).
\end{aligned}$$

Using the mean value theorem the limit as $h\to 0$ is established as before, introducing terms similar to S_1.

Therefore, $-(\partial V/\partial t)(t, x)$ exists and equals $LV(t, x)$.

COROLLARY 2.2. *Recalling the operator*

$$L = X_0(t, x)\frac{\partial}{\partial x} + \frac{1}{2}\sum_{k,l=1}^{d}\sum_{j=1}^{m} X_j^k(t, x)\, X_j^l(t, x)\frac{\partial^2}{\partial x_k\, \partial x_l}$$

we see that $V(t, x) = E[c(\xi_{0,T}(x_0)) \mid F_t^0]$ *is the solution of the backward Kolmogorov equation*

$$\frac{\partial v}{\partial t} + LV = 0$$

with terminal condition

$$\begin{aligned} V(T, x_T) &= E[c(\xi_{0,T}(x_0)) \mid F_T^0] \\ &= c(x_T). \end{aligned}$$

Remark 2.3. The situation when the coefficients X_i are not bounded can be treated by using stopping times. Equations of the form

$$\frac{\partial v}{\partial t} + LV + \phi v = 0,$$

where ϕ is a smooth, bounded function, can be treated by introducing a new coordinate ξ^{d+1}, where

$$\begin{aligned} \xi_{s,t}^{d+1}(x, y) &= y \exp\left\{\int_s^t \phi(t, \xi_{s,t}(x))\, dr\right\} \\ &= y + \int_s^t \xi_{s,r}^{d+1} \phi(r, \xi_{s,r}(x))\, dr. \end{aligned}$$

This is the Feynman–Kac form of the solution; see Kunita [5].

3. THE FORWARD EQUATION

We again work on the time interval $[0, T]$. Recall the σ-field

$$F_T^t = \sigma\{w_s - w_r : t \leqslant r \leqslant s \leqslant T\},$$

and denote by $\{F_T^t\}$, $0 \leqslant t \leqslant T$, the left-continuous complete filtration

generated by the F_T^r. Suppose $f(r)$ is a continuous $\{F_T^t\}$-adapted process, that is, $f(r)$ is F_T^r measurable. Then the Ito backward integral

$$\int_t^T f(r)\,\hat{d}w_r$$

is defined as the limit in probability of sums

$$\sum_{k=0}^{n-1} f(t_{k+1})(w_{t_{k+1}} - w_{t_k}),$$

where $t = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$, and the limit is taken as $|\Delta| = \max |t_{k+1} - t_k|$ goes to zero. Kunita [5, 6] then defines the backward stochastic differential equation

$$\hat{d}\hat{\xi}_s = -X_0(x, \hat{\xi}_s)\,ds - X_j(s, \hat{\xi}_s)\,\hat{d}w_s^j.$$

For each $t \in [0, T]$ and $x \in R^d$ this has a solution $\hat{\xi}_{t,T}(x)$ with terminal condition $\hat{\xi}_{T,T}(x) = x$. We shall suppose the coefficient vector fields X_i, $i = 0, ..., m$, satisfy the boundedness and smoothness conditions of Section 2. In fact $\hat{\xi}_{t,T}(x)$ is given by

$$\hat{\xi}_{t,T}(x) = x + \int_t^T X_0(r, \hat{\xi}_{t,T}(x))\,dr + \int_t^T X_j(r, \hat{\xi}_{r,T}(x))\,\hat{d}w_r^j. \tag{3.1}$$

Clearly $\hat{\xi}_{t,T}(x)$ is a backward $\{F_T^t\}$ semimartingale, with the time parameter t running from T to 0. The situation is the mirror image to that discussed in Sections 2 and 3, so there exists a map

$$\hat{\xi}: [0, T] \times [0, T] \times R^d \times \Omega \to R^d,$$

such that for $t \leqslant r \leqslant T$ and $x \in R^d$, $\hat{\xi}_{r,T}(x)$ is the essentially unique solution of (3.1). Furthermore, $\hat{\xi}$ is twice differentiable in x; we shall write

$$\hat{D}_{t,T} \text{ for } \frac{\partial \hat{\xi}_{t,T}}{\partial x}.$$

For a given terminal condition x_T consider the backward solution $\xi_{0,T}(x_T)$ and the quantity $c(\hat{\xi}_{0,T}(x_T))$, where c is a bounded, C^3 function on R^d, with bounded derivatives. We now consider the backward martingale

$$\hat{M}_t = E[c(\hat{\xi}_{0,T}(x_T)) \mid F_T^t].$$

By the Markov property this is equal to

$$\begin{aligned} &E[c(\hat{\xi}_{0,T}(x_T))\,|\,x], \qquad \text{where} \quad x=\hat{\xi}_{t,T}(x_T) \\ &\quad = E[c(\hat{\xi}_{0,t}(x))\,|\,x] \\ &\quad = \hat{V}(t,x), \qquad \text{say.} \end{aligned}$$

Because of the symmetrical nature of both the Markov property and the independence of Brownian increments, the analog of the argument in Section 2 for the forward flow applies to $\hat{V}(t,x)$ and shows that

$$\frac{\partial \hat{V}}{\partial t}(t,x) = L\hat{V}(t,x).$$

(Note that because time is now considered in a negative direction we consider $c(\hat{\xi}_{0,t+h}(x)) - c(\hat{\xi}_{0,t}(x))$ and so we obtain $\partial\hat{V}/\partial t$, rather than the $-\partial V/\partial t$ of Theorem 2.1.)

The initial condition for $\hat{V}$ is $\hat{V}(0, \hat{\xi}_{0,T}(x_T)) = c(\hat{\xi}_{0,T}(x_T))$, i.e., $\hat{V}(0, x_0) = c(x_0)$.

This discussion avoids the "forward equation" for the "backward process," as used by Kunita [5].

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JUMP PROCESSES

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An integration by parts formula for functions of jump process is established which follows from an ordinary integration by parts in the state space of the jump measure. The analog of the Malliavin matrix is defined; if the inverse of this matrix belongs to all $L^p(\Omega), p \geqq 1$, the jump process has a smooth density.

KEY WORDS: Malliavin calculus, jump processes, integration by parts, stochastic flows.

0. INTRODUCTION

The Malliavin calculus is a calculus of variations in function space. One of its principal applications is to establish the existence and smoothness of densities of processes defined by stochastic differential equations. Following the original work by Malliavin [10], Bismut [4], Stroock [14], and others, on continuous stochastic differential equations driven by Brownian motion, there have been papers on the Malliavin calculus for equations driven by jump processes. See, for example, the work of Bismut [5], Bichteler and Jacod [2], Bichteler, Gravereaux and Jacod [3], Leandre [9], Bass and Cranston [1] and the recent paper by Norris [13]. A central result in these papers is an integration-by-parts formula in function space which is often established by considering a perturbation of the original process and a Girsanov change of measure. However, we show below how an integration-by-parts formula can simply be obtained from classical integration by parts in the state space of the jump measure. An analogous integration-by-parts formula for continuous diffusions is established in [6]; however, the jump processes discussed in this paper require significantly different definitions and techniques. A Malliavin matrix M is introduced and if M^{-1} belongs to all the spaces $L^p, p \geqq 1$, then the

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jump process has a smooth density. Of course the crucial problem then concerns the integrability of M^{-1}; this is investigated in [3] and [5] by imposing growth conditions on the coefficients and using delicate Tauberian theorems. However, we do not discuss such problems in this paper.

1. ASSUMPTIONS AND DYNAMICS

Consider a probability space (Ω, F, P). Suppose $\mu = \mu(\omega, dt \times dz)$ is a Poisson random measure on $[0, T] \times Z$ and let $\{F_t\}, 0 \leqq t \leqq T$ be the right continuous complete filtration generated by μ. Here Z can be an open unbounded subject of R^m; in fact for simplicity we shall assume $Z = R^m$. $G(dz) = dz$ will denote Lebesgue measure on Z. We shall suppose the compensator $\nu = \nu(dt \times dz)$ of μ is of the form $dt \times G(dz) = dt \times dz$, so that the compensated measure $\tilde{\mu}$ of μ is: $\tilde{\mu}(dt \times dz) = \mu(dt \times dz) - dt \times dz$.

Dynamics 1.1

Consider a jump process $\xi_{0,t}(x_0) \in R^d$, for $0 \leqq t \leqq T$, given by

$$\xi_{0,t}(x_0) = x_0 + \int_0^t \int_Z g(\xi_{0,r^-}(x_0), z)\tilde{\mu}(dr \times dz)$$

$$+ \int_0^t \int_Z h(\xi_{0,r^-}(x_0), z)\nu(dr \times dz) + \int_0^t A(\xi_{0,r^-}(x_0))\, dr. \tag{1.1}$$

Here

$$g = (g^i),\ 1 \leqq i \leqq d, \text{ maps } R^d \times Z \to R^d$$

$$h = (h^i),\ 1 \leqq i \leqq d, \text{ maps } R^d \times Z \to R^d$$

$$A = (A^i),\ 1 \leqq i \leqq d, \text{ maps } R^d \to R^d.$$

We shall assume g, h, and A are C^∞ and have bounded derivatives in x and z of all orders. Furthermore, we suppose

$$\sup_x \left| D_x^p D_z^q g(x, z) \right| \in \bigcap_{r \geqq 1} L^r(Z)$$

and

$$\sup_x \left| D_x^p D_z^q h(x, \cdot) \right| \in L^1(Z) \cap L^2(Z) \cap L^\infty(Z)$$

for $p \geqq 1$, and that

$$\inf_x \det |I + g_x(x,z)| > 0. \tag{1.2}$$

Then from the results of Meyer [11] and Léandre [8] there exists a map

$$\phi: \Omega \times [0,T] \times R^d \to R^d$$

such that

i) for each $x \in R^d$ $\phi(\omega, t, x)$ is the unique solution of (1.1).
ii) for each ω and t the map $\phi(\omega, t, \cdot)$ is C^∞ on R^d, with derivatives of all orders which satisfy the stochastic equations obtained from (1.1) by formal differentiation.

Consequently, for example,

$$\frac{\partial \phi(\omega,t,x)}{\partial x} = \frac{\partial \xi_{0,t}(x)}{\partial x} = D_{0,t}$$

satisfies

$$dD_{0,t} = \frac{\partial g}{\partial \xi} D_{0,t^-}\, d\tilde{\mu} + \frac{\partial h}{\partial \xi} D_{0,t^-}\, dv + \frac{\partial A}{\partial \xi} D_{0,t^-}\, dt \tag{1.3}$$

with $D_{0,0} = I$, the $d \times d$ identity matrix.

The following result is similar to that of Léandre [8].

LEMMA 1.2 *Suppose V is the matrix solution of the stochastic equation*

$$V_{0,t} = I - \int_0^t \int_Z V_{0,r^-} \frac{\partial g}{\partial \xi}\, d\tilde{\mu} - \int_0^t \int_Z V_{0,r^-} \frac{\partial h}{\partial \xi}\, dv$$

$$- \int_0^t V_{0,r^-} \frac{\partial A}{\partial \xi}\, dr + \int_0^t \int_Z V_{0,r^-} \left(\frac{\partial g}{\partial \xi}\right)\left(I + \frac{\partial g}{\partial \xi}\right)^{-1} \left(\frac{\partial g}{\partial \xi}\right) d\mu. \tag{1.4}$$

Then $V_{0,t} = D_{0,t}^{-1}$ for $0 \leqq t \leqq T$.

Proof An application of the product rule shows that $d(V_{0,t} D_{0,t}) = 0$, so because $V_{0,0} D_{0,0} = I$ the result follows.

Remarks 1.3 First note that if $D_{s,u}, W_{s,u}$ are solutions of (1.3) and (1.4) with $D_{s,s} = I = V_{s,s}$, then by applying Jensen's, Burkholder's and Gronwall's inequalities we can see that

$$\sup_{s \leqq u \leqq t} \left| D_{s,u} \right| \text{ and } \sup_{s \leqq u \leqq t} \left| V_{s,u} \right| \text{ are in } L^p(\Omega) \text{ for all } p < \infty.$$

The Poisson random measure μ is a special case of an integer valued random measure. Therefore, as in (3.18) of Jacod [7], there is a set

$$D=\{(\omega,t):\mu(\omega,\{t\}\times Z)=1\}\subset\Omega\times[0,T]$$

and for each $(\omega,t)\in D$ a unique point $\beta_t(\omega)\in Z$ such that $\mu(\omega,\{t\}\times dz)$ is the Dirac measure at $\beta_t(\omega)$. Therefore,

$$\mu(\omega,dt\times dz)=\sum_{(\omega,s)D}\delta_{(s,\beta_s(\omega))}(dt\times dz).$$

Consider the solution flow $\phi(\omega,t,x_0)=\xi_{0,t}(x_0)$ of (1.1). If μ, and so ξ, has a jump at time t, the magnitude of the jump of ξ is $g(\xi_{0,t^-}(x_0),z)$, where $z=\beta_t(\omega)$. Write $x=\xi_{0,t^-}(x_0)$. Then by the uniqueness of the solution of (1.1) we have the following flow and semigroup properties:

$$\xi_{0,T}(x_0)=\xi_{t^-,T}(x)=\xi_{t,T}(\xi_{0,t}(x_0))$$

$$=\xi_{t,T}(x+g(x,z)I_D).$$

Considering the Jacobians, therefore, by differentiating and using the chain rule we see $\xi_{0,t}(x_0)=\xi_{0,t^-}(x_0)+g(\xi_{0,t^-}(x_0),z)I_D$, so

$$D_{0,t}=(I+g_x(\xi_{0,t^-}(x_0),z)I_D)D_{0,t^-}$$

and

$$D_{0,t}^{-1}=D_{0,t^-}^{-1}(I-(I+g_x(\xi_{0,t^-}(x_0)z))^{-1}g_x(\xi_{0,t^-}(x_0),z)I_D \qquad (1.5)$$

Also, with $D_{t^-,T}=\partial\xi_{t^-,T}/\partial x$,

$$D_{0,T}=D_{t^-,T}D_{0,t^-}$$

$$=D_{t,T}(I+g_xI_D)D_{0,t^-}$$

so

$$D_{t,T}=D_{0,T}D_{0,t^-}^{-1}(I+g_xI_D)^{-1}$$

$$=D_{0,T}D_{0,t^-}^{-1}(I-(I+g_x)^{-1}g_xI_D).$$

This identity indicates why condition (1.2) is necessary. Similarly, writing

$$y=\xi_{0,t}(x_0)$$

$$\xi_{0,T}(x_0)=\xi_{t,T}(y)$$

so

$$D_{0,T} = D_{t,T} D_{0,t}$$

and

$$D_{t,T} = D_{0,T} D_{0,t}^{-1}. \tag{1.6}$$

If we consider the Eq. (1.1) for $\xi_{0,t}(x_0)$ and the equation (1.3) for its Jacobian $D_{0,t}$ as a single system, the coefficients are not bounded. Following Norris [12] we introduce a class of "lower triangular" coefficients. See also Stroock, [15].

DEFINITION 1.4 For positive integers α, d, $d_1, \ldots, d_k$, with $d = d_1 + \cdots + d_k$ write $S_\alpha(d_1, \ldots, d_k)$ for the set of $X \in C(R^d \times Z, R^d)$ of the form

$$X(x, z) = \begin{pmatrix} X^{(1)}(x^1, z) \\ X^{(2)}(x^1, x^2, z) \\ \vdots \\ X^{(k)}(x^1, x^2, \ldots, x^k, z) \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix}.$$

$z \in R^m$, where R^d is identified with $R^{d_1} \times \cdots \times R^{d_k}$, $x^j \in R^{d_j}$ and the X satisfy

$$\|X\|_{S_{(\alpha,N)}} = \sup_{x \in R^d} \sup_{0 \le p+q \le N} \frac{|D_x^p D_z^q X(x, z)|}{(1 + |x|^\alpha)} \vee \sup_{1 \le j \le k} |D_j X^{(j)}(x, z)| \in L^1(Z) \cap L^\infty(Z)$$

for all positive integers N.

Write $S(d_1, \ldots, d_k) = \bigcup_\alpha S_\alpha(d_1, \ldots, d_k)$.

For coefficients not involving z a similar space $\hat{S}(d_1, \ldots, d_k)$ can be defined as in [6].

Remarks 1.5 Note (1.1) and (1.3) can be considered as a single system with the coefficients which involve z belonging to $S(d, d^2)$ and the coefficients A, $(\partial A/\partial \xi)D$, which are independent of z, belonging to $\hat{S}(d, d^2)$. The proof of Norris can then be adapted to prove the following result:

THEOREM 1.6 *Suppose* g, h, $S_\alpha(d_1, \ldots, d_k)$, $A \in \hat{S}_\alpha(d_1, \ldots, d_k)$ and $\inf_x \det |I + g_x| > 0$.

Then there is a map $\phi: \Omega \times [0, T] \times [0, T] \times R^d \to R^d$ such that

i) for $0 \le s \le t \le T$ and $x \in R^d$ $\phi(\omega, s, t, x)$ is the essentially unique solution of the equation

$$dx_t = g\, d\tilde{\mu} + h\, dv + A\, dt \tag{1.7}$$

with $x_s = x$.

ii) for each ω, s, t the map $\phi(\omega, s, t, \cdot)$ is C^∞ in x with derivatives of all orders satisfying equations derived from (1.7) by formal differentiation.

iii) $\sup_{|x| \le R} E[\sup_{s \le u \le t} |D^n \phi(\omega, s, u, x)|^p]$

$\le C(p, s, t, R, n, N, d_1, \ldots, d_k, \alpha, \|g\|_{S(\alpha, N)}, \|h\|_{S(\alpha, N)}, \|A\|_{S(\alpha, N)})$,

where D^n denotes any mixed partial operator in $\partial/\partial x_1, \ldots, \partial/\partial x_d$ of order less than n.

Proof This is a technical result whose proof, using smooth truncations of the coefficients, following that of Norris [12]. It is of interest to note that condition (1.2)

$$\inf_x \det|I+g_x|>0$$

for the system (1.7) implies the corresponding condition for the enlarged system:

$$dx_t = g\,d\tilde{\mu} + h\,dv + A\,dt$$

$$dD_{0,t} = \frac{\partial g}{\partial \xi} D_{0,t}\,d\tilde{\mu} + \frac{\partial h}{\partial \xi} D_{0,t}\,dv + \frac{\partial A}{\partial \xi} D_{0,t}\,dt. \tag{1.8}$$

The function corresponding to g, that is the coefficient of $\tilde{\mu}$, in (1.8) is

$$\bar{g} = \begin{pmatrix} g \\ \dfrac{\partial g}{\partial \xi}\otimes D \end{pmatrix}.$$

and the state space now consists of the variables x and $D_{0,t}$. Recall $\partial g/\partial \xi$ is another notation for g_x, the gradient of g, so

$$\bar{g} = \begin{pmatrix} g \\ g_x \otimes D \end{pmatrix}.$$

Forming the Jacobian of $\bar{g}$ with respect to x and D we obtain an operator we can formally denote by

$$\begin{pmatrix} g_x & 0 \\ g_{xx} D \otimes D & g_x \end{pmatrix}$$

and, working in coordinates, we see $\det|I+\bar{g}_x| = (\det|I+g_x|^3)$. Therefore, $\inf_x \det|I+\bar{g}_x|>0$.

2. INTEGRATION BY PARTS

In this section we shall establish an integration by parts formula.

ASSUMPTIONS 2.1 Suppose $g, h, A \in S_x(d_1,\dots,d_k)$,

$$\lim_{|z|\to\infty} D_z^q g = 0 \quad \text{for} \quad q \geqq 1, \tag{2.1}$$

$x_0 \in R^d$, and that conditions (1.2) are satisfied. Consider the unique solution $\xi_{0,t}(x_0)$ of the equation

$$\xi_{0,t}(x_0) = x_0 + \int_0^t \int_Z g(\xi_{0,r^-}(x_0), z)\,\tilde{\mu}(dr \times dz)$$
$$+ \int_0^t \int_Z h(\xi_{0,r^-}(x_0), z)\,v(dr \times dz) + \int_0^t A(\xi_{0,r^-}(x_0))\,dr \tag{2.2}$$

for $0 \leqq t \leqq T$.

Suppose $F:[0,T]\times R^d\to R$ is a bounded C^2 function. Then the differentiation rule gives

$$
\begin{aligned}
F(t,\xi_{0,t}(x_0)) = {} & F(0,x_0)+\int_0^t \frac{\partial F}{\partial r}(r,\xi_{0,r}(x_0))\,dr \\
& +\int_0^t\int_Z \frac{\partial F}{\partial \xi}(r,\xi_{0,r^-}(x_0))g(\xi_{0,r^-}(x_0),z)\,d\tilde{\mu} \\
& +\int_0^t\int_Z \frac{\partial F}{\partial \xi}(r,\xi_{0,r^-}(x_0))h(\xi_{0,r^-}(x_0),z)\,d\nu \\
& +\int_0^t \frac{\partial F}{\partial \xi}(r,\xi_{0,r^-}(x_0))A(\xi_{0,r^-}(x_0))\,dr \\
& +\int_0^t\int_Z (F(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z))-F(r^-,\xi_{0,r^-}(x_0)) \\
& -\frac{\partial F}{\partial \xi}(r^-,\xi_{0,r^-}(x_0))g(\xi_{0,r^-}(x_0),z)\,d\mu \\
= {} & F(0,x_0)+\int_0^t\left(\frac{\partial F}{\partial r}+\frac{\partial F}{\partial \xi}A\right)dr+\int_0^t\int_Z \frac{\partial F}{\partial \xi}h\,d\nu \\
& +\int_0^t\int_Z (F(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z))-F(r^-,\xi_{0,r^-}(x_0))\,d\tilde{\mu} \\
& +\int_0^t\int_Z (F(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z))-F(r^-,\xi_{0,r^-}(x_0)) \\
& -\frac{\partial F}{\partial \xi}(r^-,\xi_{0,r^-}(x_0))g(\xi_{0,r^-}(x_0),z)\,d\nu. \qquad (2.3)
\end{aligned}
$$

Consider a bounded C^1 function $c:R^d\to R$ with a bounded derivative c_ξ.

The random variable $c(\xi_{0,T}(x_0))$ is F_T measurable and for $0\leqq t\leqq T$ we can consider the martingale

$$M_t=E[c(\xi_{0,T}(x_0))\,|\,F_t].$$

Writing $x=\xi_{0,t}(x_0)$, because the process ξ is a Markov, we have

$$M_t=E_{t,x}[c(\xi_{0,T}(x_0)]$$

$$= E[c(\xi_{t,T}(x))]$$

$$= V(t, x), \quad \text{say}.$$

By differentiating with respect to x inside the expectation we see

$$\frac{\partial V}{\partial x} = E\left[c_\xi(\xi_{t,T}(x))\frac{\partial \xi_{t,T}}{\partial x}\right]$$

$$= E[c_\xi(\xi_{0,T}(x_0))D_{0,T} \mid F_t]D_{0,t}^{-1} \tag{2.4}$$

using (1.6).

LEMMA 2.2 *$V(t, x)$ is differentiable in t.*

Proof Consider the function $\phi(x) = c(\xi_{t,T}(x))$. Using (2.3):

$$\phi(\xi_{t-h,t}(x)) - \phi(x) = \int_{t-h}^{t} A\frac{\partial \phi}{\partial x}dr + \int_{t-h}^{t}\int_Z h\frac{\partial \phi}{\partial x}dv$$

$$+ \int_{t-h}^{t}\int_Z (\phi(\xi_{t-h,r^-}(x) + g(\xi_{t-h,r^-}(x), z)) - \phi(\xi_{t-h,r^-}(x)))\,d\tilde{\mu}$$

$$+ \int_{t-h}^{t}\int_Z (\phi(\xi_{t-h,r^-}(x) + g(\xi_{t-h,r^-}(x), z)) - \phi(\xi_{t-h,r^-}(x))$$

$$- g(\xi_{t-h,r^-}(x), z)\frac{\partial \phi}{\partial \xi}(\xi_{t-h,r^-}(x))\,dv.$$

Taking the expected value, dividing by h and letting $h \to 0$ we see

$$-\frac{\partial V}{\partial t} = A\frac{\partial V}{\partial x} + \int_Z h\frac{\partial V}{\partial x}dz$$

$$+ \int_Z \left(V(t-, x + g(x, z)) - V(t, x) - g(x, z)\frac{\partial V}{\partial x}\right)dz.$$

Because the martingale $V(t, x) = E[c(\xi_{0,T}(x_0)) \mid F_t] = E[c(\xi_{t,T}(x))]$ is differentiable in t and x we have the following martingale representation result.

PROPOSITION 2.3 *Write $y = \xi_{0,t}(x_0)$. With $V(t, y) = E[c(\xi_{t,T}(y))] = E[c(\xi_{0,T}(x_0)) \mid F_t]$*

$$V(t,\xi_{0,t}(x_0)) = V(0,x_0)$$

$$+\int_0^t\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z)) - V(r^-,\xi_{0,r^-}(x_0)))\,d\tilde{\mu}.$$

Proof We have seen above that V is differentiable in t and y. Therefore, writing down the differentiation rule (2.3) for V

$$V(t,\xi_{0,t}(x_0)) = V(0,x_0) + \int_0^t \left(\frac{\partial V}{\partial r} + A\frac{\partial V}{\partial \xi}\right)dr + \int_0^t\int_Z h\frac{\partial V}{\partial \xi}\,dv$$

$$+\int_0^t\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z)) - V(r^-,\xi_{0,r^-}(x_0)))\,d\tilde{\mu}$$

$$+\int_0^t\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z))$$

$$-V(r^-,\xi_{0,r^-}(x_0)) - g(\xi_{0,r^-}(x_0),z)\frac{\partial V}{\partial \xi}(r^-,\xi_{0,r^-}(x_0))\,dv$$

$$= V(0,x_0) + \int_0^t \left(\frac{\partial V}{\partial \xi} + LV\right)dr$$

$$+\int_0^t\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z)) - V(r^-,\xi_{0,r^-}(x_0))\,d\tilde{\mu}. \quad (2.5)$$

However, from Lemma 2.2 $\partial V/\partial r + LV = 0$ and so

$$V(t,y) = V(0,x_0) + \int_0^t\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z)) - V(r^-,\xi_{0,r^-}(x_0))\,d\tilde{\mu}.$$

The integral with respect to the compensated measure $\tilde{\mu}$ is, of course, a martingale.

COROLLARY 2.5 *Note, in particular, the representation*

$$c(\xi_{0,T}(x_0)) = E[c(\xi_{0,T}(x_0))] + \int_0^T\int_Z (V(r,\xi_{0,r^-}(x_0)+g(\xi_{0,r^-}(x_0),z)$$

$$-V(r^-,\xi_{0,r^-}(x_0))\,d\tilde{\mu}.$$

NOTATION 2.6 Suppose $\lambda(\omega, t, z) = \lambda(t, z)$ is a square integrable, predictable, possibly vector-valued, process such that both $\lambda(t, z)$ and

$$l(t, z) = \frac{\partial \lambda}{\partial z} = \left(\frac{\partial \lambda}{\partial z_1}, \ldots, \frac{\partial \lambda}{\partial z_m}\right) \text{ belong to } L^1(Z \times [0, T] \times \Omega)$$

$$\text{and } \lim_{|z| \to \infty} \lambda(t, z) = 0, \text{ for almost all } \omega. \tag{2.6}$$

Consider the vector valued martingale

$$L_t = \int_0^t \int_Z l(r, z)\, d\tilde{\mu}$$

$$= \int_0^t \int_Z \left(\frac{\partial \lambda}{\partial z_1}, \ldots, \frac{\partial \lambda}{\partial z_m}\right) d\mu - \int_0^t \int_Z \left(\frac{\partial \lambda}{\partial z_1}, \ldots, \frac{\partial \lambda}{\partial z_m}\right) dv. \tag{2.7}$$

For $1 \leqq k \leqq m$

$$\int_0^t \int_Z \frac{\partial \lambda}{\partial z_k}\, dv = \int_0^t \left(\int_Z \frac{\partial \lambda}{\partial z_k}\, dz_1, \ldots, dz_k, \ldots, dz_m\right) dr = 0$$

for almost all ω by (2.6).

Therefore, $L_t = \int_0^t \int_Z l(r, z)\, d\mu$ for almost all ω.

THEOREM 2.7 *Suppose $\xi_{0,t}(x_0)$ is the solution of Eq.* (2.2) *and L_t is defined by* (2.7). *Write $x = \xi_{0,r^-}(x_0)$. Then*

$$E[c(\xi_{0,T}(x_0)) L_T]$$

$$= -E\left[c_\xi(\xi_{0,T}(x_0)) D_{0,T} \int_0^T \int_Z D_{0,r^-}^{-1} (I + g_x(x, z))^{-1} g_z(x, z) \lambda(r, z) \mu(dz, dr)\right]$$

where

$$g_z = \left(\frac{\partial g}{\partial z_1}, \ldots, \frac{\partial g}{\partial z_m}\right).$$

Proof With V given as in Proposition 2.3 and L given by (2.7) the product rule implies that

$$V_T L_T = \int_0^T V_{r^-}\, dL + \int_0^T L_{r^-}\, dV$$

$$+ \int_0^T \int_Z (V(r, x + g(x, z)) - V(r^-, x)) l(r, z)\, d\mu.$$

Taking expectations we have

$$E[c(\xi_{0,T}(x_0))L_T] = E\left[\int_0^T\int_Z V(r, x+g(x,z))l(r,z)\,dv\right]$$

$$-E\left[\int_0^T V(r^-, x)\left(\int_Z \frac{\partial\lambda}{\partial z}(r,z)\,dz\right)dr\right], \tag{2.8}$$

and the final expectation is zero by (2.6). Furthermore, because $l(r,z) = (\partial\lambda/\partial z_1, \ldots, \partial\lambda/\partial z_m)$ we can integrate the terms by parts in $z_1, \ldots, z_m$, respectively, and obtain:

$$E\left[\int_0^T\left(\int_Z V(r, x+g(x,z))l(r,z)\,dz\right)dr\right]$$

$$= -E\left[\int_0^T\int_Z \frac{\partial V}{\partial z}(r, x+g(x,z))\lambda(r,z)\,dz\,dr\right].$$

$$= -E\left[\int_0^T\int_Z \frac{\partial V}{\partial z}(r, x+g(x,z))\lambda(r,z)\,d\mu\right] = -E\left[\int_0^T\int_Z \frac{\partial V}{\partial y}(r,y)g_z(x,z)\lambda(r,z)\,d\mu\right]$$

$$\text{where} \quad y = \xi_{0,r}(x_0). \tag{2.9}$$

Here we have again used (2.6), and $\partial V/\partial z = (\partial V/\partial z_1, \ldots, \partial V/\partial z_m)$. Recall

$$V(t,y) = E[c(\xi_{0,T}(x_0)) \mid F_t] = E[c(\xi_{t,T}(y))],$$

and from (2.4)

$$\frac{\partial V}{\partial y} = E[c_\xi(\xi_{0,T}(x_0))D_{0,T} \mid F_t]D_{0,r}^{-1},$$

so

$$E\left[\int_0^T\left(\int_Z V(r, x+g(x,z))l(r,z)\,dz\right)dr\right]$$

$$= -E\left[\int_0^T\int_Z E[c_\xi(\xi_{0,T}(x_0)D_{0,T} \mid F_r]D_{0,r}^{-1}g_z(x,z)\lambda(r,z)\,d\mu\right].$$

Writing

$$X_r = E[c_\xi(\xi_{0,T}(x_0))D_{0,T} \mid F_r]$$

and

$$Y_r = \int_0^r \int_Z D_{0,s}^{-1} g_z(x_{s-}, z)\lambda(s, z)\, d\mu$$

we have

$$X_T Y_T = \int_0^T X_r\, dY_r + \int_0^T Y_{r-}\, dX_r$$

so

$$E[X_T Y_T] = E\left[\int_0^T X_r\, dY_r\right].$$

Substituting in (2.9),

$$E[c(\xi_{0,T}(x_0))L_T]$$

$$= -E\left[c_x(\xi_{0,T}(x_0))D_{0,T}\int_0^T \int_Z D_{0,r}^{-1} g_z(x, z)\lambda(r, z)\mu(dz, dr)\right]$$

$$= -E\left[c_x(\xi_{0,T}(x_0))D_{0,T}\int_0^T \int_Z D_{0,r-}^{-1}(I + g_x(x, z))^{-1} g_z(x, z)\lambda(r, z)\mu(dz, dr)\right].$$

NOTATION 2.8 Write

$$M_{s,t} = \int_s^t \int_Z D_{s,r}^{-1} (I + g_x)^{-1} g_z(x, z) g_z^*(x, z)(I + g_x)^{*-1} D_{s,r}^{*-1}\mu(dz, dr)$$

where, as above, $x = \xi_{0,r-}(x_0)$ and $*$ denotes the transpose. Then M is the Malliavin matrix for the process.

COROLLARY 2.9 *In Theorem 2.7 take $\lambda(t, z)$ to be the process*

$$g_z^*(I + g_x)^{*-1} D_{0,t-}^{*-1}.$$

Note that from (2.1) this λ satisfies condition (2.6). Then with $l(t, z) = (\partial\lambda/\partial z)(t, z)$ and $L_T = \int_0^T \int_Z l(t, z)\, d\tilde{\mu}$

$$E[c(\xi_{0,T}(x_0))L_T] = E[c_x(\xi_{0,T}(x_0))D_{0,T}M_{0,T}]. \tag{2.10}$$

Remark 2.9 Equation (2.10) can be considered as an "integration-by-parts" formula for $c(\xi_{0,T}(x_0))$.

In Theorem 2.7 we could consider an l which is just the derivative of λ in a single direction, say $l(t, z) = \partial\lambda/\partial z_1$. Then with $L_t = \int_0^t \int_Z l(t, z)\, d\tilde{\mu}$ for this component the above calculations go through, and we obtain (2.9) with $M_{0,T}$ replaced by

$$\int_0^t \int_Z D_{0,r-}^{-1}(I + g_x)^{-1} g_{z_1} g_{z_1}^* (I + g_x)^{*-1} D_{0,r-}^{*-1}\mu(dz, dr).$$

Here $g_{z_1} = \partial g/\partial z_1$. An integration-by-parts formula (2.10) is obtained which involves the derivative of g in just a single direction. However, to preserve symmetry, we prefer formula (2.10) in the form which includes the gradient g_z.

3. HIGHER DIMENSIONAL FLOWS AND BOUNDS FOR DERIVATIVES

Consider a product function

$$f(\xi_{0,T}(x_0)) = c(\xi_{0,T}(x_0))k(\xi_{0,T}(x_0))$$

satisfying the conditions of Theorem 2.7 and apply Corollary 2.9. Then

$$E[f(\xi_{0,T}(x_0))L_T] = -E[(c_\xi(\xi_{0,T}(x_0))k(\xi_{0,T}(x_0)+c(\xi_{0,T}(x_0))k_\xi(\xi_{0,T}(x_0))D_{0,T}M_{0,T}].$$

What we would like to do in (3.1) is take $k = M_{0,T}^{-1}D_{0,T}^{-1}$ so we can obtain a bound for c_ξ. However, $D_{0,T}^{-1}$ and $M_{0,T}^{-1}$ involve the past of the processes ξ, D and M. This difficulty can be overcome, as in [6], by considering augmented flows $\phi^{(n)}$, $n = 1, 2, \ldots$, defined inductively in higher dimensional spaces. Therefore 1.6 applies to all $\phi^{(n)}$, $n \geqq 1$, so that analogs of Theorem 2.7 and Corollary 2.9 may be used with $\phi^{(1)}$ instead of ϕ and $k(\phi^{(1)}) = M_{0,T}^{-1}D_{0,T}^{-1}$ to prove the following result:

THEOREM 3.1 *Suppose $\xi_{0,t}(x_0)$ is the solution of* (2.1) *and c is any smooth function with bounded derivatives. If $M_{0,T}^{-1}$ is in some $L^p(\Omega)$*

$$|E[c_\xi(\xi_{0,T}(x_0))]| \leqq K \sup_{x \in R^d} |c(x)|.$$

Remarks 3.2 As is well known (see Malliavin [10], or Stroock [13]) the inequality (3.4) implies the random variable $\xi_{0,T}(x_0)$ has a density.

To show the density of $\xi_{0,T}(x_0)$ is differentiable we must obtain bounds of the form

$$\left| E\left[\frac{\partial^\alpha}{\partial \xi^\alpha} c(\xi_{0,T}(x_0)) \right] \right| \leqq K \sup_{x \in R^d} |c(x)| \tag{3.1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index of non-negative integers and

$$\frac{\partial^\alpha c}{\partial \xi^\alpha} = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \xi_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial \xi_d^{\alpha_d}} c.$$

An argument from Fourier analysis (see [12]) shows that if (3.1) is true for all α with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqq n$ where $n \geqq d+1$ then the random variable $\xi_{0,T}(x_0)$ has a density $d(x) = d(x_1, \ldots, x_d)$ which is in $C^{n-d-1}(R^d)$.

If we proceed as above with $\phi^{(n)}$ replacing $\phi^{(1)}$ and consider successively higher derivatives of c in the above mentioned analog of Corollary 2.9, we can derive the following result:

THEOREM 3.3 *Suppose the inverse of the Malliavin matrix, $M_{0,T}^{-1}$, is in every $L^p(\Omega)$, $1 \leqq p \leqq \infty$. Then the random variable $\xi_{0,T}(x_0)$ has a density $d(x)$ which belongs to $C^\infty(R^d)$.*

Remarks 3.4 The above has been studied in detail in [6] for the continuous case, and the methods carry over to the situation here. The main question, however, concerns the existence and integrability properties of $M_{0,T}^{-1}$. Now $M^{-1} \in L^p(\Omega)$ if and only if $\det M^{-1} \in L^p(\Omega)$, and for the symmetric matrix M we have the following inequality, (see [3]):

$$(\det M)^{-p} \leqq \Gamma(p)^{-d} \int_{R^d} |x|^{d(2p-1)}(\exp(-x^* M x)\, dx.$$

To show the integral on the right is finite delicate Tauberian theorems are applied in [5] and [3]. The conditions for these to be satisfied are established by assuming g satisfies certain growth conditions. The objective of this paper is to give the simple proof of the integration-by-parts formula (2.10), which does not involve the calculus of variations in function space, and indicate how this simplifies arguments concerning the existence of densities.

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Martingale Representation and the Malliavin Calculus*

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Abstract. Using stochastic flows and the Itô differentiation rule, the integrand in the representation of a martingale as a stochastic integral is identified. By iterating this representation result a homogeneous chaos type expansion is obtained. Using the stochastic integral representation, an integration by parts formula is obtained without using any calculus of variations in function space. If the inverse of the Malliavin matrix belongs to all spaces $L^p(\Omega)$ it follows that a random variable has a smooth density.

1. Introduction

The Malliavin calculus was originally developed in a remarkable paper, [7], as a calculus of variations in function space; one of its applications is to show that under appropriate conditions diffusion processes have densities. Bismut, in [3], approached the Malliavin calculus by using stochastic flows to describe perturbations of trajectories while Stroock in, for example, [9] used more functional analytic methods. Simplifications of the Malliavin theory were provided in the papers by Bichteler and Fonken [2] and Norris [8], and in the recent book by Bell [1]. A very readable exposition can be found in the paper by Zakai [10], and a careful treatment is in the text by Ikeda and Watanabe [5]. In all these presentations function space calculus is used. A contribution of this paper is that,

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for Markov diffusions, some of the initial results of the Malliavin theory, including the "integration by parts" formula, are obtained by techniques which involve differentiation with respect to initial conditions.

The paper begins by using the theory of stochastic flows to identify the integrand in a stochastic integral. After some rearrangement this integrand is itself written in terms of a martingale which can be expressed as a stochastic integral, and by recursively repeating the representation a homogeneous chaos expansion is obtained. Using the stochastic integral representation, an integration by parts formula is then derived. If the inverse of the Malliavin matrix M belongs to all the spaces $L^p(\Omega)$ we show that a random variable has a smooth density; however, the difficult questions concerning the relationship between Hörmander's conditions on the coefficient vector fields and the integrability of M^{-1} are not discussed in this paper. This paper was presented at the Workshop on Diffusion Approximations held at the International Institute for Applied Systems Analysis, Laxenburg, Austria, in July 1987. A fuller treatment of the ideas given here can be found in [4].

2. Dynamics

Consider a stochastic differential system

$$dx_t = X_0(t, x_t)\, dt + X_i(t, x_t)\, dw_t^i. \tag{2.1}$$

Here $x \in R^d$, $0 \le t \le T$, and $w = (w^1, \dots, w^m)$ is an m-dimensional Brownian motion on (Ω, F, P). We shall assume that the coefficient vector fields X are smooth and have bounded derivatives of all orders.

From results in [5], for example, it is known that for $0 \le s \le t \le T$ and $x_s \in R^d$ there is a unique solution $\xi_{s,t}(x_s)$ of (2.1) with $\xi_{s,s}(x_s) = x_s$. Furthermore, there is a version of this solution which, almost surely, is smooth in $x_s \in R^d$.

If $x_0 \in R^d$ and $x = \xi_{0,t}(x_0)$, because the solutions of (2.1) are unique,

$$\xi_{0,T}(x_0) = \xi_{t,T}(\xi_{0,t}(x_0)) = \xi_{t,T}(x). \tag{2.2}$$

Write $D_{s,t} = \partial \xi_{s,t}/\partial x$ for the Jacobian of the map $x \to \xi_{s,t}(x)$. Then, differentiating (2.2),

$$D_{0,T} = D_{t,T} D_{0,t}.$$

Again, from [5] we know that D satisfies the equation

$$dD_{s,t} = \frac{\partial X_0}{\partial \xi} D_{s,t}\, dt + \frac{\partial X_i}{\partial \xi} D_{s,t}\, dw_t^i \tag{2.3}$$

with $D_{s,s} = I$, the $d \times d$ identity matrix.

Consider the matrix function $V_{s,t}$ defined by the stochastic differential equation

$$dV_{s,t} = -V_{s,t} \frac{\partial X_0}{\partial \xi}\, dt - V_{s,t} \frac{\partial X_i}{\partial \xi}\, dw_t^i \tag{2.4}$$

with $V_{s,s} = I$. Here

$$\bar{X}_0^j = X_0^j - \frac{1}{2}\sum_{\alpha=1}^{m}\left(\frac{\partial X_\alpha^j}{\partial \xi_k}\right) X_\alpha^k.$$

Then, see [2], $d(V_{s,t} D_{s,t}) = 0$ so

$$V_{s,t} = D_{s,t}^{-1}.$$

3. Martingale Representation

Suppose $x_0 \in R^d$ is given. Consider a smooth, bounded function c on R^d and the random variable $c(\xi_{0,T}(x_0))$. Write $\{F_t\}$ for the right continuous, complete filtration generated by $F_t = \sigma\{w_s : s \le t\}$. Because x_0 is known $\sigma\{x_s : s \le t\} \subset F_t$ and the process (x_t, w_t) is Markov. Consider the martingale

$$M_t = E[c(\xi_{0,T}(x_0)) | F_t].$$

Then by the martingale representation result

$$M_t = M_0 + \int_0^t \gamma_i(s)\, dw_s^i \tag{3.1}$$

for some predictable, square integrable process γ. However, because $\xi_{0,t}(x_0)$ is Markov, writing $x = \xi_{0,t}(x_0)$,

$$M_t = E[c(\xi_{0,T}(x_0)) | x] = E[c(\xi_{t,T}(x))] = E[c(\xi_{t,T}(x)) | F_t] = V(t, x).$$

By the chain rule $c(\xi_{t,T}(x))$ is differentiable in x. Consequently, $V(t, x)$ is differentiable in x. By considering the backward equation for $\xi_{t,T}(x)$ as in [6] we see $V(t, x)$ is differentiable in t. Therefore, applying the Itô differentiation rule to $V(t, x)$ with $x = \xi_{0,t}(x_0)$,

$$V(t, \xi_{0,t}(x_0)) = V(0, x_0) + \int_0^t \left(\frac{\partial V}{\partial s} + LV\right) ds + \int_0^t \frac{\partial V}{\partial x} \cdot X_i\, dw_s^i. \tag{3.2}$$

Here

$$L = \sum_{i=1}^{d} X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\left(\sum_{k=1}^{m} X_k^i X_k^j\right)\frac{\partial^2}{\partial x_i\, \partial x_j}.$$

However, $V(t, \xi_{0,t}(x_0)) = M_t$ so the decompositions (3.1) and (3.2) must be the same. The bounded variation term in (3.2) is, therefore, zero, i.e.,

$$\frac{\partial V}{\partial s} + LV = 0$$

and (as is well known) V is the solution of the backward Kolmogorov equation with a final condition

$$c(x_T) = V(T, x_T).$$

Equating the martingale terms in (3.1) and (3.2)

$$\gamma_i(t)=\frac{\partial V}{\partial x}\cdot X_i.$$

Differentiating inside the expectation

$$\frac{\partial V}{\partial x}=E[c_\xi(\xi_{t,T}(x))D_{t,T}|F_t] \qquad \text{(by the chain rule)}$$

$$=E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_t]D_{0,t}^{-1}.$$

So

$$\gamma_i(t)=E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_t]D_{0,t}^{-1}X_i \tag{3.3}$$

and

$$M_t=E[c(\xi_{0,T}(x_0))]+\int_0^t E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s]D_{0,s}^{-1}X_i(s,\xi_{0,s}(x_0))\,dw_s^i. \tag{3.4}$$

Remark 3.1. Note the term $E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s]$ is itself a martingale. If the representation is written down at $t=T$

$$M_T=c(\xi_{0,T}(x_0))$$

$$=E[c(\xi_{0,T}(x_0))]+\int_0^T E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s]D_{0,s}^{-1}X_i\,dw_s^i. \tag{3.5}$$

Also, the representation (3.4) holds for vector (or matrix) functions c.

If we take $c(\xi)=\xi$ to be the identity map on R^d, (3.5) gives

$$\xi_{0,T}(x_0)=E[\xi_{0,T}(x_0)]+\int_0^T E[D_{0,T}|F_s]D_{0,s}^{-1}X_i\,dw_s^i.$$

Also, if we consider (3.5) for a second smooth bounded function g and take the expected value of the product of each side, we see

$$E[c(\xi_{0,T}(x_0))g(\xi_{0,T}(x_0))]$$

$$=E[c(\xi_{0,T}(x_0))]E[g(\xi_{0,T}(x_0))]$$

$$+E\left[\sum_{i=1}^m\int_0^T E[c_\xi D_{0,T}|F_s]D_{0,s}^{-1}X_iX_i^*D_{0,s}^{*-1}E[g_\xi D_{0,T}|F_s]^*\,ds\right]. \tag{3.6}$$

Definition 3.2. The Malliavin matrix for the system (2.1) is

$$M_{s,t}=\sum_{i=1}^m\left(\int_s^t D_{s,u}^{-1}X_i(u)X_i^*(u)D_{s,u}^{*-1}\,du\right).$$

Note something resembling $M_{0,s}$ occurs in (3.6).

4. Homogeneous Chaos Expansions

Consider an enlarged system with components $\xi^{(1)}=(\xi, D)$. The stochastic differential equation for $\xi^{(1)}$ is, therefore, the system (2.1) and (2.3). The coefficients in (2.3) are no longer bounded, but following Norris [8] a sequence of "triangular" systems can be considered and the results on stochastic flows still hold. We can, therefore, consider the Jacobian $D^{(1)}$ of the system $\xi^{(1)}$ and a system $\xi^{(2)}=(\xi^{(1)}, D^{(1)})$. Proceeding in this way $\xi^{(n)}$ is a system with components $(\xi^{(n-1)}, D^{(n-1)})$. Write

$$c^{(1)}=\frac{\partial c}{\partial \xi} D_{0,T},$$

$$c^{(2)}=\frac{\partial c^{(1)}}{\partial \xi^{(1)}} D_{0,T}^{(1)}, \quad \text{etc.}$$

Equation (3.4) can then be written

$$c(\xi_{0,T}(x_0))=E[c(\xi_{0,T}(x_0))]+\int_0^T E[c^{(1)}|F_s]D_{0,s}^{-1}X_i\,dw_s^i. \tag{4.1}$$

However, $E[c^{(1)}(\xi_{0,T}^{(1)})|F_s]$ can be represented, as in Section 3, as a stochastic integral

$$E[c^{(1)}|F_s]=E[c^{(1)}]+\int_0^s E[c^{(2)}|F_{s_1}]D_{0,s_1}^{(1)-1}X_j^{(1)}(s_1)\,dw_{s_1}^j.$$

Here, $X_j^{(1)}$ is the coefficient vector field of w^j in the system defining $\xi^{(1)}$. Substituting in (4.1)

$$\begin{aligned} c(\xi_{0,T}(x_0)) &= E[c]+E[c^{(1)}]\int_0^T D_{0,s}^{-1}X_i\,dw_s^i \\ &\quad +\int_0^T\left(\int_0^s E[c^{(2)}|F_{s_1}]D_{0,s_1}^{(1)-1}X_j^{(1)}(s_1)\,dw_{s_1}^j\right)D_{0,s}^{-1}X_i\,dw_s^i. \end{aligned} \tag{4.2}$$

Now $E[c^{(2)}|F_{s_1}]$ can be expressed as a stochastic integral and the result substituted in (4.2). Proceeding in this way we obtain the homogeneous chaos expansion of the random variable $c(\xi_{0,T}(x_0))$. The repeated stochastic integrals do not involve c but only the Jacobians $D^{(k)}$ and coefficients $X^{(k)}$.

5. Integration by Parts

Lemma 5.1. *Suppose $u=(u_1,\dots,u_m)$ is a square integrable predictable process. Then*

$$E\left[c(\xi_{0,T}(x_0))\int_0^T u_i\,dw_s^i\right]=\sum_{i=1}^m E\left[c_\xi(\xi_{0,T}(x_0))D_{0,T}\int_0^T D_{0,s}^{-1}X_i(s)u_i(s)\,ds\right].$$

Proof. Consider the representation (3.5) for $c(\xi_{0,T}(x_0))$. Multiply by $\int_0^T u_t\,dw_s^i$ and, using Fubini's theorem, take the expectation. □

Corollary 5.2. *Take* $u_i(s)=(D_{0,s}^{-1}X_i(s))^*$. *Then*

$$E\left[c(\xi_{0,T}(x_0))\int_0^T (D_{0,s}^{-1}X_i(s))^*\,dw_s^i\right]=E[c_\xi(\xi_{0,T}(x_0))D_{0,T}M_{0,T}]. \tag{5.1}$$

Remark 5.3. Consider a product function $h(\xi_{0,T}(x_0))=c(\xi_{0,T}(x_0))g(\xi_{0,T}(x_0))$ and apply Corollary 5.2 to h. Then

$$E\left[(cg)(\xi_{0,T}(x_0))\int_0^T (D_{0,s}^{-1}X_i)^*\,dw_s^i\right]=E[(c_\xi g+cg_\xi)D_{0,T}M_{0,T}]. \tag{5.2}$$

We would like to take $g=M_{0,T}^{-1}D_{0,T}^{-1}$ in (5.2) so that we can obtain a bound for c_ξ. This can be done by considering, again following Norris [8], a hierarchy of stochastic systems similar to, but different from, those introduced in Section 4.

This time write $\varphi^{(0)}(w,s,t,x)=\xi_{s,t}(x)$ for the flow defined by (2.1) and $D_{s,t}^{(0)}(x)=D_{s,t}(x)$ for its Jacobian. $R_{s,t}^{(0)}=\int_s^t(D_{s,u}^{-1}X_i(u))^*\,dw_u^i$ and $M_{s,t}^{(0)}=M_{s,t}$ is the Malliavin matrix defined in (3.2). Note that $M_{s,t}$ can be considered as the predictable quadratic variation of the tensor product of $R^{(0)}$ with its adjoint, that is $M_{s,t}^{(0)}=\langle R^{(0)}\otimes R^{(0)^*}\rangle_{s,t}$.

Now consider an enlarged system $\varphi^{(1)}$ with components

$$\varphi^{(1)}=(\varphi^{(0)},D^{(0)},R^{(0)},M^{(0)}).$$

The results of Norris [8] on stochastic flows allow us to discuss the Jacobian $D^{(1)}$ of $\varphi^{(1)}$. Assume $X_i^{(1)}$ is the coefficient of w^i in the system describing $\varphi^{(1)}$, and write

$$R_{s,t}^{(1)}=\int_s^t (D_{s,u}^{(1)-1}X_i^{(1)}(u))^*\,dw_u^i,$$

$$M_{s,t}^{(1)}=\langle R^{(1)}\otimes R^{(0)^*}\rangle_{s,t}.$$

Then define

$$\varphi^{(2)}=(\varphi^{(1)},D^{(1)},R^{(1)},M^{(1)})$$

and inductively, $\varphi^{(n+1)}=(\varphi^{(n)},D^{(n)},R^{(n)},M^{(n)})$. Write ∇_n for the gradient operator in the components of $\varphi^{(n)}$. The following result is established like equation (5.2) by considering the martingale representation (3.5) of the produce cg.

Theorem 5.4. *Suppose c is a bounded C^∞ scalar function on R^d with bounded derivatives. Let g be a C^∞ possibly vector, or matrix, valued function on the state space of $\varphi^{(n)}$ such that $g(\varphi^{(n)}(0,T,x_0))$ and $\nabla_n g(\varphi^{(n)}(0,T,x_0))$ are both in some $L^p(\Omega)$. Then*

$$\begin{aligned}&E[c(\varphi^{(0)}(0,T))g(\varphi^{(n)}(0,T))\otimes R_{0,T}^{(0)}]\\&\quad=E[(\nabla_0 c)(\varphi^{(0)}(0,T))g(\varphi^{(n)}(0,T))D_{0,T}M_{0,T}]\\&\qquad+E[c(\varphi^{(0)}(0,T))(\nabla_n g)(\varphi^{(n)}(0,T))D_{0,T}^{(n)}M_{0,T}^{(n)}].\end{aligned} \tag{5.3}$$

Corollary 5.5. *Gronwall's inequality shows that D^{-1} is in all the $L^p(\Omega)$ spaces, so if $M_{0,T}^{-1}$ is in some $L^p(\Omega)$ taking $g(\varphi^{(1)}(0,T)) = M_{0,T}^{-1}D_{0,T}^{-1}$ in* (5.3)

$$E[c_\xi(\xi_{0,T}(x_0))] = E[c(\xi_{0,T}(x_0))M_{0,T}^{-1}D_{0,T}^{-1}\otimes R_{0,T}] - E[c(\xi_{0,T}(x_0))(\nabla_1 g)(D_{0,T}, M_{0,T})D_{0,T}^{(1)}M_{0,T}^{(1)}].$$

Because c is bounded we, therefore, have the following result:

Theorem 5.6. *Suppose $\xi_{0,T}(x_0)$ is the solution of* (2.1) *and c is any smooth bounded function with bounded derivatives. Then if $M_{0,T}^{-1}$ is in some $L^p(\Omega)$*

$$|E[c_\xi(\xi_{0,T}(x_0))]| \le K \sup_{x\in R^d}|c(x)|. \tag{5.4}$$

Remark 5.7. It is well known that (5.4) implies that the random variable $\xi_{0,T}(x_0)$ has a density $d(x)$. To show the density d is smooth we wish to establish inequalities of the form

$$\left|E\left[\frac{\partial^\alpha c}{\partial\xi^\alpha}(\xi_{0,T}(x_0))\right]\right| \le K \sup_{x\in R^d}|c(x)|. \tag{5.5}$$

Here

$$\frac{\partial^\alpha}{\partial\xi^\alpha} = \frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial\xi_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_d}}{\partial\xi_d^{\alpha_d}}.$$

An argument from Fourier analysis (see [8]) shows that if (5.5) is true for all α with $|\alpha| = \alpha_1 + \cdots + \alpha_d \le n$ where $n \ge d+1$, then the random variable $\xi_{0,T}(x_0)$ has a density $d(x)$ which is in $C^{n-d-1}(R^d)$.

Apply Corollary 5.5 to c_ξ rather than c so

$$E[c_{\xi\xi}(\xi_{0,T}(x_0))] = E[c_\xi(\xi_{0,T}(x_0))M_{0,T}^{-1}D_{0,T}^{-1}\otimes R_{0,T}] - E[c_\xi(\xi_{0,T}(x_0))(\nabla_1 g)(D_{0,T}, M_{0,T})D_{0,T}^{(1)}M_{0,T}^{(1)}]. \tag{5.6}$$

Consider the two terms on the right of (5.6) and write $M = M_{0,T}$, $D = D_{0,T}$, etc. Let

$$g_1(\varphi^{(1)}) = M^{-1}D^{-1}\otimes RM^{-1}D^{-1}$$

and

$$g_2(\varphi^{(2)}) = (\nabla_1 g)(D, M)D^{(1)}M^{(1)}M^{-1}D^{-1}.$$

Applying Theorem 5.4 to cg_1 and cg_2,

$$E[c(\xi_{0,T}(x_0))g_1(\varphi^{(1)})\otimes R] = E[c_\xi(\xi_{0,T}(x_0))M^{-1}D^{-1}\otimes R] + E[c(\xi_{0,T}(x_0))(\nabla_2 g_1)(\varphi^{(2)})D^{(2)}M^{(2)}] \tag{5.7}$$

and

$$E[c(\xi_{0,T}(x_0))g_2(\varphi^{(2)})\otimes R] = E[c_\xi(\xi_{0,T}(x_0))(\nabla_1 g)(D, M)D^{(1)}M^{(1)}] + E[c(\xi_{0,T}(x_0))(\nabla_3 g_2)(\varphi^{(3)})D^{(3)}M^{(3)}]. \tag{5.8}$$

Using (5.7) and (5.8) the terms on the right of (5.6) can be replaced by terms involving c. This procedure can be iterated using Theorem 5.4 and the following result established:

Theorem 5.8. *Suppose M^{-1} is in all spaces $L^p(\Omega)$, $1 \le p < \infty$. Then the random variable $\xi_{0,T}(x_0)$ has a smooth density.*

The remaining questions concern the existence and integrability properties of $M_{0,T}^{-1}$. These have been carefully studied (see [5] or [8], for example). In fact $M_{0,T}^{-1}$ is in $L^p(\Omega)$ for all p, $1 \le p < \infty$, if the following condition of Hörmander is satisfied:

Condition 5.9. The vector space $V(x_0)$ generated by the coefficient vector fields $X_1, \ldots, X_m$ and the brackets $[X_i, X_j]$, $0 \le i, j \le m$, $[X_i, [X_j, X_k]]$, $0 \le i, j, k \le m$, etc., evaluated at $x_0 \in R^d$, is the whole of R^d.

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The Conditional Adjoint Process

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Summary

The adjoint process and minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. Using stochastic flows the variation in the cost resulting from a change in an optimal control can be computed explicitly. The technical difficulty is to justify the differentiation.

1. INTRODUCTION.

Using stochastic flows we calculate below the change in the cost due to a 'strong' variation of an optimal control. Differentiating this quantity enables us to identify the adjoint, or co-state variable, and give a partially observed minimum principle. If the drift coefficient is differentiable in the control variable the related result of Bensoussan [2] follows from our theorem. Full details will appear in [1]. The method appears simpler than that employed in Haussman [4].

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2. DYNAMICAL EQUATIONS.

Suppose the state of a stochastic system is described by the equation

$$d\xi_t = f(t, \xi_t, u)dt + g(t, \xi_t)dw_t,$$

$$\xi_t \in R^d, \qquad \xi_0 = x_0, \qquad 0 \le t \le T. \tag{2.1}$$

The control variable u will take values in a compact subset U of some Euclidean space R^k. We shall assume

A_1: $x_0 \in R^d$ is given.

A_2: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t, x), continuously differentiable in x for each (t, u) and

$$(1 + |x|)^{-1}|f(t, x, u)| + |f_x(t, x, u)| \le K_1.$$

A_3: $g : [0,T] \times R^d \to R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some K_2:

$$|g(t, x)| + |g_x(t, x)| \le K_2.$$

The observation process is defined by

$$dy_t = h(\xi_t)dt + d\nu_t \tag{2.2}$$

$$y_t \in R^m, \qquad y_0 = 0, \qquad 0 \le t \le T.$$

In (2.1) and (2.2) $w = (w^1, \ldots, w^n)$ and $\nu = (\nu^1, \ldots, \nu^m)$ are independent Brownian notions defined on a probability space (Ω, F, P).

Furthermore, we assume

A_4: $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x and

$$|h(t, x)| + |h_x(t, x)| \le K_3.$$

REMARK 2.1. These hypotheses can be weakened to those discussed by Haussman [4]. See [1].

Write $\hat{P}$ for the Wiener measure on $C([0,T], R^n)$ and μ for the Wiener measure on $C([0,T], R^m)$.

$$\Omega = C([0,T], R^n) \times C([0,T], R^m)$$

and the coordinate functions in Ω will be denoted (x_t, y_t). Wiener measure P on Ω is

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

DEFINITION 2.2. *$Y = \{Y_t\}$ will be the right continuous, complete filtration on $C([0,T], R^m)$ generated by*

$$Y_t^0 = \sigma\{y_s : s \leq t\}.$$

The set of admissible control functions $\underline{U}$ will be the Y-predictable functions defined on $[0,T] \times C([0,T], R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$, $\xi^u_{s,t}(x)$ will denote the strong solution of (2.1) corresponding to u with $\xi^u_{s,s} = x$.

Define

$$Z^u_{s,t}(x) = \exp\Big(\int_s^t h(\xi^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 dr\Big). \tag{2.3}$$

Note a version of Z defined for every trajectory y can be obtained by integrating the stochastic integral in the exponential by parts.

If a new probability measure P^u defined on Ω by putting

$$\frac{dP^u}{dP} = Z^u_{0,T}(x_0),$$

under P^u $(\xi^u_{0,t}(x_0), y_t)$ is a solution of the system (2.1) and (2.2). That is, under P^u, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

Because of hypothesis A_4, for $0 \leq t \leq T$ easy applications of Burkholder's and Gronwall's inequalities show that

$$E[(Z^u_{0,t}(x_0))^p] < \infty \tag{2.4}$$

for all $u \in \underline{U}$ and all p, $1 \leq p < \infty$.

COST 2.3. We shall suppose the cost is purely terminal and equals

$$c(\xi^u_{0,T}(x_0))$$

where c is a bounded, differentiable function. If control $u \in \underline{U}$ is used the expected cost is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

With respect to P, under which y_t is a Brownian motion

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.5}$$

A control $u^* \in \underline{U}$ is optimal if

$$J(u^*) \leq J(u)$$

for all $u \in \underline{U}$. We shall suppose there is an optimal control u^*.

3. FLOWS.

For $u \in \underline{U}$ and $x \in R^d$ consider the strong solution

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)dr + \int_s^t g(r, \xi^u_{s,r}(x))dw_r. \tag{3.1}$$

We wish to consider the behaviour of $\xi^u_{s,t}(x)$ for each trajectory y of the observation process. In fact the results of Bismut [3] and Kunita [6] extend and show the map

$$\xi^u_{s,t} : R^d \to R^d$$

is, almost surely, a diffeomorphism for each $y \in C([0,T], R^m)$.

Write

$$\|\xi^u(x_0)\|_t = \sup_{0 \leq s \leq t} |\xi^u_{0,s}(x_0)|.$$

Then, using Gronwall's and Jensen's inequalities, for any p, $1 \leq p < \infty$

$$\|\xi^u(x_0)\|^p_T \leq C\Big(1 + |x_0|^p + \Big|\int_0^T g(r, \xi^u_{0,r}(x_0))dw_r\Big|^p\Big)$$

almost surely, for some constant C.

Using A_3 and Burkholder's inequality

$$\|\xi^u(x_0)\|_T \in L^p \quad \text{for} \quad 1 \leq p < \infty.$$

Suppose u^* is an optimal control, and write

$$\xi^*_{s,t}(\cdot) \quad \text{for} \quad \xi^{u^*}_{s,t}(\cdot).$$

The Jacobian $\frac{\partial \xi^*_{s,t}}{\partial x}$ is the matrix solution C_t of the equation

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t dt + \sum_{i=1}^n g^{(i)}_x(t, \xi^*_{s,t}(x))C_t dw^i_t. \tag{3.2}$$

with $C_s = I$.

Here $g^{(i)}$ is the i^{th} column of g and I is the $n \times n$ identity matrix. Writing $\|C\|_T = \sup_{0 \le s \le t} |C_s|$ and using Burkholder's, Jensen's and Gronwall's inequalities we see $\|C\|_T \in L^p$, $1 \le p < \infty$.

Consider the matrix valued process D defined by

$$D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r)dr$$
$$- \sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x))dw^i_r + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x)))^2 dr \tag{3.3}$$

Then as in [5] or [6] $d(D_t C_t) = 0$ and $D_s C_s = I$ so

$$D_t = C_t^{-1} = \left(\frac{\partial \xi^*_{s,t}}{\partial x}\right)^{-1}.$$

Furthermore, $\|D\|_t \in L^p$, $1 \le p < \infty$.

Suppose $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw^i_r$ is a d-dimensional semimartingale. Bismut [3] shows one can consider the process $\xi^*_{s,t}(z_t)$ and in fact:

$$\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r, \xi^*_{s,r}(z_r), u^*_r)$$
$$+ \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r) \frac{\partial \xi^*_{s,r}}{\partial x} H_i$$
$$+ \frac{1}{2} \sum_{i=2}^n \frac{\partial^2 \xi^*_{s,r}}{\partial x^2}(H_i, H_i) \Big) dr$$
$$+ \int_s^t \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big) dw^i_r. \tag{3.4}$$

DEFINITION 3.1. *For $s \in [0, T]$, $h > 0$ such that $0 \le s < s + h \le T$, for any $\tilde{u} \in \underline{U}$, and $A \in Y_s$ consider a 'strong' variation u of u^* defined by*

$$u(t, w) = \begin{cases} u^*(t, w) & \text{if } (t, w) \notin [s, s+h] \times A \\ \tilde{u}(t, w) & \text{if } (t, w) \in [s, s+h] \times A. \end{cases}$$

THEOREM 3.2. *For any strong variation u of u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr. \tag{3.5}$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

PROOF. We shall substitute in (3.4), (noting $H_i = 0$ for all i). Therefore,

$$\begin{aligned}\xi^*_{s,t}(z_t) &= x + \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr \\ &+ \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1}(f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr \\ &+ \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r.\end{aligned}$$

The solution of (3.1) is unique, so $\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x)$. Note $u(t) = u^*(t)$ if $t > s+h$ so $z_t = z_{s+h}$ if $t > s+h$ and

$$\begin{aligned}\xi^*_{s,t}(z_t) &= \xi^*_{s,t}(z_{s+h}) \\ &= \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)). \end{aligned} \tag{3.6}$$

4. THE EXPONENTIAL DENSITY.

Consider the $(d+1)$-dimensional system

$$\begin{aligned}\xi^*_{s,t}(x) &= x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)dr + \int_s^t g(r, \xi^*_{s,r}(x))dw_r \\ Z^*_{s,t}(x,z) &= z + \int_s^t Z^*_{s,r}(x,z)h(\xi^*_{s,r}(x))'dy_r. \end{aligned}\tag{4.1}$$

That is, we are considering an augmented flow (ξ, Z) in R^{d+1} in which Z^* has a variable initial condition $z \in R$. Note:

$$Z^*_{s,t}(x,z) = zZ^*_{s,t}(x).$$

The map $(x,z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x,z))$ is, almost surely, a diffeomorphism of R^{d+1}. Clearly,

$$\frac{\partial \xi^*_{s,t}}{\partial z} = 0, \qquad \frac{\partial f}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial z} = 0.$$

The Jacobian of this augmented map is represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \frac{\partial \xi^*_{s,t}}{\partial x} & 0 \\ \frac{\partial Z^*_{s,t}}{\partial x} & \frac{\partial Z^*_{s,t}}{\partial z} \end{pmatrix}.$$

In particular, from (4.1), for $1 \le i \le d$

$$\frac{\partial Z^*_{s,t}}{\partial x_i} = \sum_{j=1}^{m} \int_s^t \Big(Z^*_{s,r}(x,z) \sum_{k=1}^{n} \frac{\partial h^j}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}}{\partial x_i} + h^j\left(\xi^*_{s,r}(x)\right) \frac{\partial Z^*_{s,r}}{\partial x_i}\Big) dy^j_r. \tag{4.2}$$

We are interested in solutions of (4.1) and (4.2) only when $z = 1$, so as above we write

$$Z^*_{s,t}(x) \quad \text{for} \quad Z^*_{s,t}(x,1) \quad \text{etc.}$$

LEMMA 4.1.
$$\frac{\partial Z^*_{s,t}}{\partial x} = Z^*_{s,t}(x)\Big(\int_s^t h_x(\xi^*_{s,t}(x)) \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big)$$
where, as in (2.2), $d\nu_t = dy_t - h(\xi^*_{s,t}(x))dt$.

PROOF. From (4.2)

$$\frac{\partial Z^*_{s,t}}{\partial x} = \int_s^t \Big(\frac{\partial Z^*_{s,r}}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}}{\partial x}\Big) dy_r. \tag{4.3}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\Big(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big).$$

Then

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$

and the product rule gives

$$L_{s,t}(x) = \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$
$$+ \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} dy_r.$$

Consequently, $L_{s,t}(x)$ is also a solution of (4.3), so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^*_{s,t}}{\partial x}.$$

LEMMA 4.2. *If z_t is as defined in (3.5)*

$$Z^*_{s,t}(z_t) = Z^u_{s,t}(x).$$

PROOF.

$$Z^u_{s,t}(x) = 1 + \int_s^t Z^u_{s,r}(x) h'(\xi^u_{s,r}(x)) dy_r. \tag{4.4}$$

Applying (3.4) to $Z^*_{s,t}(z_t)$ we see:

$$Z^*_{s,t}(z_r) = 1 + \int_s^t Z^*_{s,r}(z_r)h'(\xi^*_{s,r}(z_r))dy_r$$

$$= 1 + \int_s^t Z^*_{s,r}(z_r)h'(\xi^u_{s,r}(x))dy_r$$

by Theorem 3.2. However, (4.4) has a unique solution so

$$Z^*_{s,t}(z_r) = Z^v_{s,t}(x).$$

Again, note that for $t > s + h$

$$Z^*_{s,t}(z_t) = Z^*_{s,t}(z_{s+h}). \tag{4.5}$$

5. THE ADJOINT PROCESS.

u^* will be an optimal control and u a perturbation of u^* as in Definition 3.1. Again write

$$x = \xi^*_{0,s}(x_0).$$

The minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0)c(\xi^*_{0,T}(x_0))]$$

$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(x)c(\xi^*_{s,T}(x))].$$

Also,

$$J(u) = E[Z^*_{0,s}(x_0)Z^u_{s,T}(x)c(\xi^u_{s,T}(x))]$$

$$= E[Z^*_{0,s}(x_0)Z^*_{s,T}(z_{s+h})c(\xi^*_{s,T}(z_{s+h}))]$$

by (3.6) and (4.5). Recall $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable almost surely, with continuous and uniformly integrable derivatives. Consequently, writing

$$\Gamma(s, z_r) = Z^*_{0,s}(x_0)Z^*_{s,T}(z_r)\Big\{c_\xi(\xi^*_{s,T}(z_r))\,\frac{\partial \xi^*_{s,T}}{\partial x}\,(z_r)$$

$$+ c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r))\,\frac{\partial \xi^*_{s,\sigma}}{\partial x}\,(z_r)d\nu_\sigma\Big)\Big\}\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}$$

for $s \le r \le s+h$, we have

$$J(u) - J(u^*) = E[Z^*_{0,s}(x_0)\{Z^*_{s,t}(z_{s+h})c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x)c(\xi^*_{s,T}(x))\}]$$
$$= E\Big[\int_s^{s+h} \Gamma(s,z_r)(f(r,\xi^*_{s,r}(z_r),u_r) - f(r,\xi^*_{s,r}(x),u^*_r))dr\Big]. \tag{5.1}$$

This formula describes the change in the expected cost arising from the perturbation u of the optimal control. However, $J(u) \ge J(u^*)$ for all $u \in \underline{U}$ so the right hand side of (5.1) is non-negative for all $h > 0$. We wish to divide by $h > 0$ and let $h \to 0$. This requires some careful arguments using the uniform boundedness of the random variables and the monotone class theorem. It can be shown that there is a set $S \subset [0,T]$ of zero Lebesgue measure such that if $s \notin S$

$$E[\Gamma(s,x)(f(s,\xi^*_{0,s}(x_0),u) - f(s,\xi^*_{0,s}(x_0),u^*_s))I_A] \ge 0 \tag{5.2}$$

for any $u \in U$ and $A \in Y_s$.

Details of this argument can be found in [1]. Define

$$p_s(x) = E^*\Big[c_\xi(\xi^*_{0,T}(x_0))\,\frac{\partial \xi^*_{s,T}}{\partial x}(x)$$
$$+ c(\xi^*_{0,T}(x_0))\Big(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0))\,\frac{\partial \xi^*_{s,\sigma}}{\partial x}\,(x)d\nu_\sigma\Big)\Big|\, Y_s \vee \{x\}\Big]$$

where $x = \xi^*_{0,s}(x_0)$ and E^* is the expectation under $P^* = P^{u^*}$.

In (5.2) we have established the following:

THEOREM 5.1. *$p_s(x)$ is the adjoint process for the partially observed optimal control problem. That is, if $u^* \in \underline{U}$ is optimal there is a set $S \subset [0,T]$ of zero Lebesgue measure such that for $s \notin S$*

$$E^*[p_s(x)f(s,x,u^*)\mid Y_s] \ge E^*[p_s(x)f(s,x,u)\mid Y_s] \quad \text{a.s.} \tag{5.3}$$

so the optimal control u^* almost surely minimizes the conditional Hamiltonian.

If $x = \xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under P^*, and if f is differentiable in u, (5.3) implies

$$\sum_{i=1}^k (u_i(s) - u^*_i(s)) \int_{R^d} \Gamma(s,x)\,\frac{\partial f}{\partial u_i}\,(s,x,u^*)q_s(x)dx \ge 0.$$

This is the result of Bensoussan [2].

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ORDINARY DIFFERENTIAL EQUATIONS AND FLOWS

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1. INTRODUCTION.

The theory of stochastic flows was first developed in the works of Kunita [5] and Bismut [2]; they have been used to discuss, for example, stochastic control [1] and the Malliavin calculus [4]. However, some of the corresponding ideas concerning deterministic flows do not appear so well known to those working in ordinary differential equations, although they are probably familiar in terms of vector fields and their pull-backs to differential topologists.

2. DYNAMICS.

Consider an ordinary differential equation

$$d\xi_t = f(t, \xi)dt \tag{2.1}$$

where $\xi \in R^d$ and $t \geq 0$.

For simplicity we shall suppose $f : [0, \infty) \times R^d \to R^d$ is C^∞ and of linear growth. Write $\xi^f_{s,t}(x) = \xi_{s,t}(x)$ for the unique solution of (2.1) which is such that $\xi_{s,s}(x) = x$, i.e.,

$$\xi_{s,t}(x) = x + \int_s^t f(u, \xi_{s,u}(x))du. \tag{2.2}$$

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Then it is known the map

$$x \to \xi_{s,t}(x)$$

is differentiable. Write

$$D_{s,t}(x) = D_{s,t} = \frac{\partial \xi_{s,t}(x)}{\partial x}$$

for its derivative. Then D is the solution of the linearized equation

$$dD = f_\xi(t, \xi) D dt \tag{2.3}$$

with $D_{s,s} = I$, the $d \times d$ identity matrix.

In integrated form this is

$$D_{s,t} = I + \int_s^t f_\xi(u, \xi_{s,u}(x)) D_{s,u} \, du. \tag{2.4}$$

Consider the matrix $V_{s,t}$ defined by

$$V_{s,t} = I - \int_s^t V_{s,u} f_\xi(u, \xi_{s,u}(x)) du \tag{2.5}$$

i.e., $dV = -V f_\xi(t, \xi) dt$. Then $V_{s,s} D_{s,s} = I$ and from (2.3) and (2.5)

$$\begin{aligned} d(VD) &= (dV)D + V(dD) \\ &= -V f_\xi D dt + V f_\xi D dt \\ &= 0 \end{aligned}$$

so $V_{s,t} D_{s,t} = I$ for all $t \geq 0$. Consequently,

$$V_{s,t} = D_{s,t}^{-1} = \left(\frac{\partial \xi_{s,t}(x)}{\partial x} \right)^{-1}.$$

Suppose z_t, for $t \geq s$, is some differentiable path in R^d. Rather than the map $x \to \xi_{s,t}(x)$ we can consider the composite map

$$z_t \to \xi_{s,t}(z_t).$$

Note t occurs twice on the right, so taking the derivative we have:

$$\xi_{s,t}(z_t) = z_s + \int_s^t f(u, \xi_{s,u}(z_u))du$$
$$+ \int_s^t \frac{\partial \xi_{s,u}}{\partial x}(z_u)dz_u. \tag{2.6}$$

Suppose $g : R^d \to R^d$ is a second function like f. Consider the equation

$$d\xi_t = g(t, \xi)dt. \tag{2.7}$$

The unique solution of (2.7) starting at $x \in R^d$ at time s will be denoted by $\xi^g_{s,t}(x)$, so

$$\xi^g_{s,t}(x) = x + \int_s^t g(u, \xi^g_{s,u}(x))du. \tag{2.8}$$

We then have the following formula for ξ^g.

THEOREM 2.1. *$\xi^g_{s,t}(x) = \xi^f_{s,t}(z_t)$ where*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^f_{s,u}(z_u)}{\partial x}\right)^{-1} (g(u, \xi^f_{s,u}(z_u)) - f(u, \xi^f_{s,u}(z_u)))du.$$

PROOF. As above, we shall write $\xi_{s,t}(x)$ for $\xi^f_{s,t}(t)$ etc. Then, substituting this z_t in (2.6):

$$\xi_{s,t}(z_t) = x + \int_s^t f(u, \xi_{s,u}(z_u))du$$
$$+ \int_s^t \left(\frac{\partial \xi_{s,u}(z_u)}{\partial x}\right) \left(\frac{\partial \xi_{s,u}(z_u)}{\partial x}\right)^{-1} (g(u, \xi_{s,u}(z_u)) - f(u, \xi_{s,u}(z_u)))du$$
$$= x + \int_s^t g(u, \xi_{s,u}(z_u))du.$$

However, (2.8) has a unique solution so

$$\xi^g_{s,t}(x) = \xi_{s,t}(z_t).$$

This result is particularly useful in optimal control when one wishes to compute the variation in the cost due to a perturbation of an optimal control. See [1] and [4].

3. BACKWARD EQUATIONS.

Consider the solution $\xi_{s,t}(x)$ of (2.2). If $F : R^d \to R$ is a C^2 function, by the chain rule

$$F(\xi_{s,t}(x)) = F(x) + \int_s^t F_\xi(\xi_{s,u}(x)) f(u, \xi_{s,u}(x)) du. \tag{3.1}$$

Consider a partition $\pi = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of $[s,t]$ and write $|\pi| = \max_i |t_{i+1} - t_i|$. Then one can also write

$$F(\xi_{s,t}(x)) - F(x) = \sum_{k=0}^{n-1} \big(F(\xi_{t_k,t}(x)) - F(\xi_{t_{k+1},t}(x))\big). \tag{3.2}$$

If $s \le r \le t$, by the uniqueness of the solutions of (2.2), we have the semigroup property of the flows

$$\xi_{s,t}(x) = \xi_{r,t}(\xi_{s,r}(x)). \tag{3.3}$$

For the k-th term in the sum (3.2):

$$\begin{aligned} F(\xi_{t_k,t}(x)) - F(\xi_{t_{k+1},t}(x)) &= F(\xi_{t_{k+1},t}(\xi_{t_k,t_{k+1}}(x))) - F(\xi_{t_{k+1},t}(x)) \\ &= F(\xi_{t_{k+1},t}(y)) - F(\xi_{t_{k+1},t}(x)) \end{aligned}$$

where

$$y = \xi_{t_k,t_{k+1}}(x) = x + \int_{t_k}^{t_{k+1}} f(u, \xi_{t_k,u}(x)) du.$$

By the mean value theorem this difference is:

$$= F_\xi(\xi_{t_{k+1},t}(z)) \frac{\partial \xi_{t_{k+1}}(z)}{\partial x} \int_{t_k}^{t_{k+1}} f(u, \xi_{t_k,u}(x)) du$$

where z is some point on the line joining x and y. Using the differentiability of the functions involved, this can be written as

$$= F_\xi(\xi_{t_{k+1},t}(x)) \frac{\partial \xi_{t_{k+1},t}(x)}{\partial x} \cdot f(t_{k+1}, x)(t_{k+1} - t_k) + R_k$$

where $|R_k| \le C(t_{k+1} - t_k)^2 \le C|\pi|(t_{k+1} - t_k)$ for some uniform bound C. The left side of (3.2) does not involve the partition π; considering partitions such that $|\pi| \to 0$ we have the following 'backward' equation:

$$F(\xi_{s,t}(x)) = F(x) + \int_s^t F_\xi(\xi_{u,t}(x)) \frac{\partial \xi_{u,t}(x)}{\partial x} f(u,x)du. \tag{3.4}$$

Clearly (3.4) holds for vector functions F. The solution $\xi_{s,t}(x)$ of (2.1) is the 'forward' flow from x. Taking $F : R^d \to R^d$ to be the identity, $F(\xi) = \xi$, we have from (3.4) the following 'backward' equation for the 'forward' flow:

$$\xi_{s,t}(x) = x + \int_s^t \frac{\partial \xi_{u,t}(x)}{\partial x} f(u,x)du. \tag{3.5}$$

By analogy with (2.1) we can also consider the following 'backward' equation:

$$-d\hat{\xi}_{s,t} = -f(s, \hat{\xi}_{s,t})ds \tag{3.6}$$

with a terminal condition x at time t. That is, we consider the 'backward' process $\hat{\xi}_{s,t}(x)$ defined by

$$\hat{\xi}_{s,t}(x) = x - \int_s^t f(u, \hat{\xi}_{u,t}(x))du \tag{3.7}$$

so $\hat{\xi}_{t,t}(x) = x$. Again, the map $x \to \hat{\xi}_{s,t}(x)$ is differentiable with a derivative $\frac{\partial \hat{\xi}_{s,t}(x)}{\partial x}$. For a smooth bounded function $F : R^d \to R$ the chain rule, (in the s variable), gives:

$$F(\hat{\xi}_{s,t}(x)) = F(x) - \int_s^t F_\xi(\hat{\xi}_{u,t}(x)) f(u, \hat{\xi}_{u,t}(x))du. \tag{3.8}$$

This is the 'backward' equation for the 'backward' flow. Similarly to (3.4) we can establish the 'forward' equation for the 'backward' flow:

$$F(\hat{\xi}_{s,t}(x)) = F(x) - \int_s^t F_\xi(\hat{\xi}_{s,u}(x)) \frac{\partial \hat{\xi}_{s,u}(x)}{\partial x} f(u,x)du. \tag{3.9}$$

In particular, taking F to be the identity map on R^d, $F(\xi) = \xi$,

$$\hat{\xi}_{s,t}(x) = x - \int_s^t \frac{\partial \hat{\xi}_{s,u}(x)}{\partial x} f(u,x)du. \tag{3.10}$$

Approximation arguments, for example, would tell us that

$$\xi_{s,t}\left(\hat{\xi}_{s,t}(x)\right) = \hat{\xi}_{s,t}\left(\xi_{s,t}(x)\right) = x.$$

However, let us proceed as follows: Differentiating (3.7) the backward equation for the backward derivative $\frac{\partial \hat{\xi}_{s,t}}{\partial x} = \hat{D}_{s,t}$ is

$$\hat{D}_{s,t} = I - \int_s^t f_\xi(u, \hat{\xi}_{u,t}(x))\hat{D}_{u,t}\, du. \tag{3.11}$$

Consider the augmented flow defined by the pair of equations (3.7), (3.11), with the 'variable' terminal condition $\hat{D}_{t,t} = D$. That is, consider $\hat{\xi}_{s,t}(x)$ defined by (3.7) and $\hat{D}_{s,t}(x, D)$ defined by

$$\hat{D}_{s,t}(x, D) = D - \int_s^t f_\xi(u, \hat{\xi}_{u,t}(x))\hat{D}_{u,t}(x, D)du. \tag{3.12}$$

Note equations (3.11) and (3.12) are linear, so that, if $\hat{D}_{s,t} = \hat{D}_{s,t}(x, I) = \hat{D}_{s,t}(x)$ is the unique solution of (3.11), then $\hat{D}_{s,t}(x)D = \hat{D}_{s,t}(x, D)$ is the unique solution of (3.12). Therefore,

$$\frac{\partial \hat{D}_{s,t}(x, D)}{\partial D} = \hat{D}_{s,t}(x). \tag{3.13}$$

Applying (3.10) to $\hat{D}_{s,t}(x, D)$, and noting derivatives in both variables x and D are involved, the forward equation for $\hat{D}_{s,t}(x, D)$ is

$$\begin{aligned}\hat{D}_{s,t}(x, D) = D &- \int_s^t \frac{\partial \hat{D}_{s,u}(x, D)}{\partial x} f(u, x)du \\ &- \int_s^t \frac{\partial \hat{D}_{s,u}(x, D)}{\partial D} f_\xi(u, x)du.\end{aligned}$$

Putting $D = I$ and substituting (3.13), the forward equation for $\hat{D}_{s,t}(x, I) = \hat{D}_{s,t}(x)$ is:

$$\hat{D}_{s,t}(x) = I - \int_s^t \frac{\partial \hat{D}_{s,u}(x)}{\partial x} f(u, x)du - \int_s^t \hat{D}_{s,u}(x) f_\xi(u, x)du.$$

Using (2.6) we have

$$\begin{aligned}\hat{D}_{s,t}\left(\xi_{s,t}(x)\right) = I &- \int_s^t \frac{\partial \hat{D}_{s,u}}{\partial x}\left(\xi_{s,u}(x)\right) f(u, \xi_{s,u}(x))du \\ &- \int_s^t \hat{D}_{s,u}\left(\xi_{s,u}(x)\right) f_\xi(u, \xi_{s,u}(x))du + \int_s^t \frac{\partial \hat{D}_{s,u}}{\partial x}\left(\xi_{s,u}(x)\right) f(u, \xi_{s,u}(x))du \\ = I &- \int_s^t \hat{D}_{s,u}\left(\xi_{s,u}(x)\right) f_\xi(u, \xi_{s,u}(x))du.\end{aligned}$$

Therefore, $\hat{D}_{s,t}(\xi_{s,t}(x))$ satisfies the same equation, (2.5), as that defining $V_{s,t}$. By uniqueness, we have the following result:

THEOREM 3.1. $\hat{D}_{s,t}(\xi_{s,t}(x)) = V_{s,t} = D_{s,t}^{-1}(x)$.

Similar arguments in the opposite time direction applied to $D_{s,t}(x)$, defined by (2.4), show that

$$D_{s,t}(\hat{\xi}_{s,t}(x)) = \hat{D}_{s,t}^{-1}(x). \tag{3.14}$$

Using flows the following result can be established:

THEOREM 3.2. $\xi_{s,t}(\hat{\xi}_{s,t}(x)) = x$.

PROOF. Applying (2.6), and using the forward equation (3.10) for $\hat{\xi}_{s,t}(x)$

$$\begin{aligned}\xi_{s,t}(\hat{\xi}_{s,t}(x)) &= x + \int_s^t f(u, \xi_{s,u}(\hat{\xi}_{s,u}(x)))du \\ &\quad - \int_s^t D_{s,u}(\hat{\xi}_{s,u}(x))\hat{D}_{s,u}(x)f(u,x)du \\ &= x + \int_s^t (f(u, \xi_{s,u}(\hat{\xi}_{s,u}(x))) - f(u,x))du \end{aligned} \tag{3.15}$$

by (3.14). However, $\xi_{s,t}(\hat{\xi}_{s,t}(x)) = x$ is a solution of (3.15), so the result follows because (3.15) has a unique solution.

Similar arguments again show $\hat{\xi}_{s,t}(\xi_{s,t}(x)) = x$.

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The variational principle for optimal control of diffusions with partial information

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Abstract: Strong variations are described for the ε-optimal control of a class of control problems for systems described by stochastic diffusion equations. The differentiation process developed identifies the adjoint process.

Keywords: Stochastic control; stochastic differential equations; diffusion equation; minimum principle.

1. Introduction

In an earlier paper [4] the authors have applied a powerful non-convex minimization result established by Ekeland [3] to derive a local approximate minimum principle for a partially observed control problem. A metric is introduced on the space of admissible controls which measures the distance between two controls. A strong variation of an ε-optimal control leads to an inequality, where on the right hand side this metric appears. As it can be expressed by a measure on the set where the perturbed ε-optimal control and the control itself differ, the inequality may be differentiated to obtain the local minimum principle.

In [1] theorems by Bismut [2] and Kunita [6] on stochastic flows are applied to give an easy and explicit calculation of the change in the cost due to the strong variation of an optimal control. These results are used here to describe the strong variation of an ε-optimal control. Using the result of Ekeland [3] the resulting inequality can be differentiated, so giving a completely new proof of the results in [4]. Furthermore, the differentiation process identifies the adjoint process; this is the main contribution of this paper. The underlying model here is the one considered in [1] and differs from the one in [4]: the drift coefficients f, h and the diffusion coefficient g in both signal and observation process depend only on the current state of the system, and not on the whole past as in [4]; the controls, however, need not be Markov. Furthermore, we impose differentiability assumptions on these coefficients and on the cost functional.

To apply Ekeland's result we show that the cost function of the control problem described in Section 2 is continuous when the control functions are topologized using the metric

$$d(u_1, u_2) = \tilde{P}(\{(t, x) \in [0,1] \times C([0,1], \mathbb{R}^m) \mid u_1(t, x) \neq u_2(t, x)\}). \tag{1.1}$$

Here $\tilde{P}$ is the product of Lebesgue measure on [0,1] and Wiener measure on $C([0,1], \mathbb{R}^m)$.

Then Ekeland's result [3] tells us that for any $\varepsilon > 0$ there is a control function u_ε such that

$$J(u_\varepsilon) \leq \inf J(u) + \varepsilon \tag{1.2}$$

and for all other control functions u,

$$J(u) \geq J(u_\varepsilon) - \varepsilon d(u_\varepsilon, u). \tag{1.3}$$

That is, u_ε minimizes the functional

$$J_\varepsilon(u) = J(u) + \varepsilon d(u, u_\varepsilon). \tag{1.4}$$

It is then shown that u_ε minimizes to within ε the conditional expectation of a certain Hamiltonian $H(s, x, \xi^\varepsilon, p, u)$, where ξ^ε is the output of the system for control u_ε, and p is an adjoint process which will be derived explicitly.

2. The control problem

We shall treat the same control problem as in [1]. Let us quickly sketch its basic properties.

Suppose the state of the system is described by the stochastic differential equation

$$\mathrm{d}\xi_t = f(t, \xi_t, u)\,\mathrm{d}t + g(t, \xi_t)\,\mathrm{d}w_t, \quad \xi_t \in R^d, \quad \xi_0 = x_0 \in R^d, \quad 0 \le t \le 1. \tag{2.1}$$

We shall make the following assumptions:

A1. x_0 is given; if x_0 is a random variable and P_0 its distribution the situation when $\int |x|^q \,\mathrm{d}P_0 < \infty$ for some $q > n+1$ can be treated by including an extra integration with respect to P_0.

A2. $f. [0,1] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ is Borel measurable, continuous on the compact metric space U for each (t, x), continuously differentiable in x with derivative f_x, and for some constant K_1,

$$(1 + |x|)^{-1} |f(t, x, u)| + |f_x(t, x, u)| \le K_1.$$

A3. $g. [0,1] \times \mathbb{R}^d \to \mathbb{R}^n \otimes \mathbb{R}^n$ is a matrix valued function, Borel measurable, continuously differentiable in x with derivative g_x, and for some constant K_2,

$$|g(t, x)| + |g_x(t, x)| \le K_2.$$

The observation process is given by

$$\mathrm{d}y_t = h(\xi_t)\,\mathrm{d}t + \mathrm{d}v_t, \quad y_t \in h^m, \quad y_0 = 0, \quad 0 \le t \le 1. \tag{2.2}$$

In the above equations $w = (w^1, \dots, w^n)$ and $v = (v^1, \dots, v^m)$ are independent Brownian notions. We also assume:

A4. $h : \mathbb{R}^d \to \mathbb{R}^m$ is Borel measurable, continuously differentiable in x, and for some constant K_3,

$$|h(t, x)| + |h_x(t, x)| \le K_3.$$

As noted in [1] these conditions can be relaxed.

Let $\hat{P}$ denote Wiener measure on $C([0,1], \mathbb{R}^n)$ and μ denote Wiener measure on $C([0,1], \mathbb{R}^m)$. Consider the basic space $\Omega = C([0,1], \mathbb{R}^n) \times C([0,1], \mathbb{R}^m)$ and define Wiener measure P on Ω by

$$P(\mathrm{d}w, \mathrm{d}y) = \hat{P}(\mathrm{d}w)\mu(\mathrm{d}y).$$

The control parameter u will take values in a compact subset U of some Euclidean space R^k. Let $Y = \{Y_t\}$ be the right continuous, complete filtration generated by $(y_s, s \le t)$. Then an admissible control is a mapping

$$u : [0,1] \times C([0,1], \mathbb{R}^m) \to U$$

which is Y-predictable. Write $\mathcal{U}$ for the set of admissible controls.

For $u \in \mathcal{U}$ and $x \in \mathbb{R}^d$ write $\xi^u_{s,t}(x)$ for the strong solution of (2.1) corresponding to control u and such that $\xi^u_{s,s}(x) = x$. Put

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))' \, \mathrm{d}y_r - \tfrac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 \, \mathrm{d}r\right)$$

and define a new probability measure P^u on Ω by $\mathrm{d}P^u/\mathrm{d}P = Z^u_{0,1}(x)$. Then under $P^u(\xi^u_{0,t}(x_0), y_t)$ is a solution of (2.1) and (2.2), that is $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

We shall consider a terminal cost given by some continuously differentiable, bounded function $c(\xi^u_{0,1}(x_0))$. The cost for admissible control $u \in \mathcal{U}$ then is

$$J(u) = E_u[c(\xi^u_{0,1}(x_0))] = E[Z^u_{0,1}(x_0)c(\xi_{0,1}(x_0))]. \tag{2.3}$$

It is shown in [4] that $J(u)$ is continuous on $\mathcal{U}$, when $\mathcal{U}$ is given the topology induced by the metric

$$d(u_1, u_2) = \tilde{P}(\{(t, y) \in [0,1] \times C([0,1], \mathbb{R}^m) \mid u_1(t, y) \neq u_2(t, y)\}).$$

Furthermore, it is shown there that $(\mathcal{U}, d)$ is a complete metric space.

3. Stochastic flows

It is not known whether there is always an optimal control for the problem described in Section 2. However, there is always an ε-optimal control for any $\varepsilon > 0$. Consider such an ε-optimal control $u_\varepsilon(t, \eta)$ satisfying (1.2) and (1.3). From our observations in [1] there exists a countable, dense subset of $\mathcal{U}$, and for the strong variations below we only take elements from this subset.

Let u_h be a strong variation of u_ε defined by:

$$u_h(t, y) = \begin{cases} u(t, y) & \text{if } s \leq t < s+h \leq 1, \\ u_\varepsilon(t, y) & \text{otherwise.} \end{cases}$$

For notational convenience drop the subscript h in u_h, and let $(\xi^\varepsilon_{s,t})$ be the strong solution of the dynamics (2.1) with input u_ε, and $(\xi^u_{s,t})$ the solution for input u, i.e.,

$$\xi^{u_\varepsilon}_{s,t} = \xi^\varepsilon_{s,t}(x) = x + \int_s^t f(r, \xi^\varepsilon_{s,r}(x), u_\varepsilon(r)) \, \mathrm{d}r + \int_s^t g(r, \xi^\varepsilon_{s,r}(x)) \, \mathrm{d}w_r \tag{3.1}$$

and

$$\xi^u_{s,t} = x + \int_s^t f(r, \xi^u_{s,r}(x), u(r)) \, \mathrm{d}r + \int_s^t g(r, \xi^u_{s,r}(x)) \, \mathrm{d}w_r. \tag{3.2}$$

It is well known (see [6]) that there are versions of these solutions such that $\xi^\varepsilon_{s,t}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is almost surely a diffeomorphism, with a Jacobian $C_t = (\partial\xi^\varepsilon_{s,t}/\partial x)(x)$ which is the solution of the equation

$$\mathrm{d}C_t = f_x(t, \xi^\varepsilon_{s,t}(x), u^\varepsilon)C_t \, \mathrm{d}t + \sum_{i=1}^d g^i_x(t, \xi^\varepsilon_{s,t}(x))C_t \, \mathrm{d}w^i_t, \quad C_s = I \ (n \times n \text{ identity mat\ldots}) \tag{3.3}$$

Here f_x denotes the partial derivative of f with respect to x and g^i_x denotes the gradient of the i-th column of g.

The inverse of the Jacobian $D_t = (\partial\xi^\varepsilon_{s,t}(x)/\partial x)^{-1}$, $C_tD_t = I$ for $t \geq s$, also satisfies a stochastic differential matrix equation, namely

$$\begin{aligned} D_t = I &- \int_s^t D_r f_x(r, \xi^\varepsilon_{s,r}(x), u^\varepsilon_r)' \, \mathrm{d}r - \sum_{i=1}^d \int_s^t D_r g^i_x(r, \xi^\varepsilon_{s,r})' \, \mathrm{d}w^i_r \\ &- \sum_{i=1}^d \int_s^t D_r(g^i_x(r, \xi^\varepsilon_{s,r}(x))')^2 \, \mathrm{d}r. \end{aligned} \tag{3.4}$$

Next we describe a change of drift induced by considering a certain semimartingale (z_t) as the initial condition in the dynamics equation. This result is due to Bismut [2]. It allows us to consider $\xi^u_{s,t}(x)$ as the unperturbed process $\xi^\varepsilon_{s,t}$ with initial condition (z_t).

Theorem 3.1. *For the perturbation u of u_ε consider the semimartingale*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^\varepsilon_{s,r}(z_r)}{\partial x}\right)^{-1} \left(f(r, \xi^\varepsilon_{s,r}(z_r), u_r) - f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r)\right) \mathrm{d}r. \tag{3.5}$$

Then the process $\xi^\varepsilon_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

Proof. From the results of Bismut ([2], Theorem 3.1) $\xi^\varepsilon_{s,t}(z_t)$ is a semimartingale with the representation

$$\begin{aligned} \xi^\varepsilon_{s,t}(z_t) &= x + \int_s^t f(r, \xi^\varepsilon_{s,t}(z_r), u^\varepsilon_r)\, \mathrm{d}r \\ &\quad + \int_s^t \left(\frac{\partial \xi^\varepsilon_{s,r}(x_r)}{\partial x}\right)\left(\frac{\partial \xi^\varepsilon_{s,r}(z_r)}{\partial x}\right)^{-1} \left(f(r, \xi^\varepsilon_{s,r}(z_r), u_r) - f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r)\right) \mathrm{d}r \\ &\quad + \int_s^t g(r, \xi^\varepsilon_{s,r}(z_r))\, \mathrm{d}w_r. \end{aligned} \tag{3.6}$$

That is,

$$\xi^\varepsilon_{s,t}(z_t) = x + \int_s^t f(r, \xi^\varepsilon_{s,r}(z_r), u_r)\, \mathrm{d}r + \int_s^t g(r, \xi^\varepsilon_{s,r}(z_r))\, \mathrm{d}w_r.$$

As the solution $(\xi^u_{s,t}(x))$ of (3.2) is unique it follows tha $\xi^\varepsilon_{s,t}(z_t) = \xi^u_{s,t}(x)$. Note that for $t > s + h$, $\xi^\varepsilon_{s,t}(z_t) = \xi^\varepsilon_{s,t}(z_{s+h}) = \xi^\varepsilon_{s+h,t}(\xi^u_{s,s+h}(x))$, because u equals u^ε $t > s + h$ and so $z_t = z_{s+h}$ for $t > s + h$. □

Next we study the augmented flow $(\xi^\varepsilon_{s,t}(x), Z^\varepsilon_{s,t}(x, z))$, i.e., the diffeomorphism on $\mathbb{R}^{d+1}$ given by

$$\begin{aligned} \xi^\varepsilon_{s,t}(x) &= x + \int_s^t f(r, \xi^\varepsilon_{s,r}(x), u_r)\, \mathrm{d}r + \int_s^t g(r, \xi^\varepsilon_{s,r}(x))\, \mathrm{d}w_r, \\ Z^\varepsilon_{s,t}(x) &= z + \int_s^t Z^\varepsilon_{s,r}(x, z) h(\xi^\varepsilon_{s,r}(x))^1\, \mathrm{d}y_r. \end{aligned} \tag{3.7}$$

To justify this discussion note there is a strong solution of (3.7) defined for every $u \in U$ and $y \in C([0,1], R^n)$, because the stochastic integral in the exponential defining Z can be integrated by parts.

As $\partial \xi^\varepsilon_{s,t}(x)/\partial z = 0$, $\partial f/\partial z = 0$, $\partial g/\partial z = 0$, the Jacobian for the augmented flow may be represented as

$$C^+_t = \begin{pmatrix} \dfrac{\partial \xi^\varepsilon_{s,t}(x)}{x} & 0 \\ \dfrac{\partial Z^\varepsilon_{s,t}(x)}{\partial x} & \dfrac{\partial Z^\varepsilon_{s,t}(x, z)}{\partial z} \end{pmatrix} \tag{3.8}$$

and the Jacobian of $Z^\varepsilon_{s,t}$ satisfies

$$\frac{\partial Z^\varepsilon_{s,t}(x, z)}{\partial x_i} = \sum_{j=1}^m \int_s^t Z^\varepsilon_{s,r}(x, z) \frac{\partial h^j(\xi^\varepsilon_{s,r}(x))}{\partial \xi} \frac{\partial \xi^{\varepsilon,k}_{s,r}(x)}{\partial x_i} + h^j(\xi^\varepsilon_{s,r}(x)) \frac{\partial Z^\varepsilon_{s,r}(x, z)}{\partial x_i}\, \mathrm{d}y^j_r$$

for $1 \le i \le d$. Here summation takes place over double indices.

Obviously, we are only interested in the solution of (3.9) for the case $z = 1$. Write $Z^\varepsilon_{s,t}(x) = Z^\varepsilon_{s,t}(x, 1)$ and from [1] we cite the following result.

Lemma 3.2. (i) *The following representation holds:*

$$\frac{\partial Z^\varepsilon_{s,t}(x)}{\partial x} = Z^\varepsilon_{s,t}(x)\left(\int_s^t h_x(\xi^\varepsilon_{s,r}(x)\frac{\partial \xi^\varepsilon_{s,r}(x)}{\partial x}\,\mathrm{d}w_r\right)$$

and (ii) $Z^\varepsilon_{s,t}(z_t) = Z^u_{s,t}(x)$, *where* (z_t) *is the semimartingale defined in* (3.5).

Again note here for $t > s+h$,

$$Z^\varepsilon_{s,t}(z_t) = Z^\varepsilon_{s,t}(z_{s+h}).$$

4. The minimum principle

The cost associated with u^ε is given by

$$\begin{aligned} J(u^\varepsilon) &= E[Z^\varepsilon_{0,1}(x_0)c(\xi^\varepsilon_{0,1}(x_0))] \\ &= E[Z^\varepsilon_{0,s}(x_0)Z^\varepsilon_{s,1}(x)c(\xi^\varepsilon_{s,1}(x))] \quad (\text{where } x = \xi^\varepsilon_{0,s}(x_0)), \end{aligned}$$

and the cost corresponding to the perturbed control is

$$J(u) = E[Z^\varepsilon_{0,s}(x_0)Z^u_{s,1}(x)c(\xi^u_{s,1}(x))] = E[Z^\varepsilon_{0,s}(x_0)Z^\varepsilon_{s,1}(z_{s+h})c(\xi^\varepsilon_{s,1}(z_{s+h}))].$$

Now $Z^\varepsilon_{s,1}(\cdot)$ and $c(\xi^\varepsilon_{s,1}(\cdot))$ are continuously differentiable, so we can compute the difference between these costs as:

$$J(u^\varepsilon) - J(u) = E\left[\int_s^{s+h} \Lambda(s, r, z_r)(f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r) - f(r, \xi^\varepsilon_{s,r}(z_r), u_r))\right] \mathrm{d}r$$

where

$$\begin{aligned} \Lambda(s, r, z_r) = Z^\varepsilon_{0,s}Z^\varepsilon_{s,1}(z_r)\Bigg\{ & c_x(\xi^\varepsilon_{s,1}(z_r))\frac{\partial \xi^\varepsilon_{s,1}(z_r)}{\partial x} \\ & + c(\xi^\varepsilon_{s,1}(z_r))\left(\int_s^T h_x(\xi^\varepsilon_{s,\sigma}(z_r))\frac{\partial \xi^\varepsilon_{s,\sigma}(z_r)}{\partial x}\,\mathrm{d}\nu_\sigma\right)\Bigg\}\left(\frac{\partial \xi^\varepsilon_{s,r}(z_r)}{\partial x}\right)^{-1}. \end{aligned}$$

Then

$$\begin{aligned} J(u^\varepsilon) - J(u) = & \int_s^{s+h} E[\Lambda(s, r, z_r) - \Lambda(s, r, x)](f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r) - f(r, \xi^\varepsilon_{s,r}(z_r), u_r))\,\mathrm{d}r \\ & + \int_s^{s+h} E[\Lambda(s, r, x) - \Lambda(r, r, x)](f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r) - f(r, \xi^\varepsilon_{s,r}, u_r))\,\mathrm{d}r \\ & + \int_s^{s+h} E[\Lambda(r, r, x)(f(r, \xi^\varepsilon_{s,r}(z_r), u^\varepsilon_r) - f(r, \xi^\varepsilon_{s,r}(z_r), u_r) \\ & \qquad - f(r, \xi^\varepsilon_{s,r}(x), u^\varepsilon_r) + f(r, \xi^\varepsilon_{s,r}(x), u_r))]\,\mathrm{d}r \\ & + \int_s^{s+h} E[\Lambda(r, r, x)(f(r, \xi^\varepsilon_{s,r}(x_0), u^\varepsilon_r) - f(r, \xi^\varepsilon_{0,r}(x_0), u_r))]\,\mathrm{d}r \\ = & I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.} \end{aligned}$$

Now

$$|I_1(h)| \le K_1 \int_s^{s+h} E\left[|\Lambda(s, r, z_r) - \Lambda(s, r, x)| \left(1 + \|\xi^u_{0,\cdot}(x_0)\|_{s+h}\right)\right] \mathrm{d}r$$

$$\le K_1 h \sup_{s \le r \le s+h} E\left[|\Lambda(s, r, z_r) - \Lambda(s, r, x)| \left(1 + \|\xi^u_{0,\cdot}(x_o)\|_{s+h}\right)\right],$$

$$|I_2(h)| \le K_2 h \sup_{s \le r \le s+h} E\left[|\Lambda(s, r, x) - \Lambda(r, r, x)| \left(1 + \|\xi^u_{0,\cdot}(x_0)\|_{s+h}\right)\right],$$

$$|I_3(h)| \le K_3 h \sup_{s \le r \le s+h} E\left[|\Lambda(r, r, x)| \|\xi^\varepsilon_{s,r}(z_r) - \xi^\varepsilon_{s,r}(x)\|_{s+h}\right].$$

The differences on the right hand side of the three above inequalities are uniformly bounded in some L^p and they converge to zero a.s. when h goes to zero. So, the differences converge to zero in L_p-norm. Then

$$\lim_{u \to 0} h^{-1} I_k(h) = 0 \quad \text{for } k = 1, 2, 3.$$

From our remark in the beginning of Section 3, u is an element in a countable dense subset of $\mathscr{U}$. As

$$\int_0^s E\left[\Lambda(r, x)\left(f(r, \xi^\varepsilon_{0,r}(x_0), u^\varepsilon_r) - f(r, \xi^\varepsilon_{0,r}, u_r)\right)\right] \mathrm{d}r$$

is finite, there are null sets $N(u^\varepsilon)$ and $N(u)$, such that for $s \notin N(u) \cup N(u^\varepsilon)$ the above expression is differentiable, and

$$\lim_{h \to 0+} h^{-1}(J(u^\varepsilon) - J(u)) = E\left[\Lambda(s, s, x)\left(f(s, \xi^\varepsilon_{0,s}(x_0), u^\varepsilon_s) - f(s, \xi^\varepsilon_{0,s}(x_0), u_s)\right)\right].$$

Therefore, from (1.3),

$$E\left[\Lambda(s, s, x)\left(f(s, \xi^\varepsilon_{0,s}(x_0), u^\varepsilon_s) - f(s, \xi^\varepsilon_{0,s}(x_0), u_s)\right)\right] \le \lim_{h \to 0+} \left(\varepsilon d(u_\varepsilon, u)\frac{1}{h}\right) \le \varepsilon \tag{4.1}$$

because $d(u, u^\varepsilon) \le h$. Using a general result of Kushner [7], the same relation holds for the conditional expectation $E[\cdot \mid Y_s]$.

Write

$$p_s(x) = E_\varepsilon\left[c_\xi(\xi^\varepsilon_{0,1}(x_0))\frac{\partial \xi^\varepsilon_{s,1}(x)}{\partial x} + c(\xi^\varepsilon_{0,1}(x_0))\left(\int_s^1 h_\xi(\xi^\varepsilon_{o,\sigma}(x_0))\frac{\partial \xi^\varepsilon_{s,\sigma}(x)}{\partial x}\, \mathrm{d}\nu_\sigma\right) \mid Y_s \vee \{x\}\right],$$

where E_ε denotes expectation under P^{u_ε}.

Then substituting in (4.1) we arrive at our minimum principle:

Theorem 4.1. *Let u^ε be an ε-optimal control and let u be any control in $\mathscr{U}$. Then there is a Lebesgue null set N, such that for $s \notin N$ the following inequality holds*:

$$E_\varepsilon\left[p_s(x)\left(f(s, \xi^\varepsilon_{0,s}(x_0), u^\varepsilon_s) - f(s, \xi^\varepsilon_{0,s}(x_0), u_s)\right) \mid Y_s\right] \le \varepsilon \quad \text{a.e.} \quad \square$$

Remark 4.2. (i) Note that if an optimal control u^* exists, and if u^ε is replaced by u^* in Theorem 4.1, then we obtain the minimum principle in [1].

(ii) If $J(u)$ is Gateaux differentiable we may deduce from Theorem 4.1 the ε-minimum principle in [5].

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A PROOF OF THE MINIMUM PRINCIPLE USING FLOWS

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A PROOF OF THE MINIMUM PRINCIPLE USING FLOWS

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Presented in honour of the memory of Zdzisław Opial.

1. Introduction. The theory of stochastic flows was developed by Kunita, [4], and Bismut, [2]. The concepts and techniques from this theory have been used to discuss the Malliavin calculus [1], and have again returned to deterministic flows, [3]. In this paper we show how concepts from the theory of deterministic flows can be used to provide an elegant proof of the Pontrjagin minimum principle.

2. The Dynamics of the Optimal Control Problem as a Flow.

Consider the control problem

(1) $\dot{x} = f(t, x, u)$, $0 \leq t \leq T$, $x(t) \in R^d$, $x(0) = x_0$, with $u(\cdot)$ measurable, $u(t) \in U \subset R^m$.

We assume that f is sufficiently well-behaved that:

(a) solutions to initial value problems are unique in $[0, T] \times R^d$ and each solution extends to $[0, T]$;

(b) the solution to the IVP (1), $x(s) = x_0$, is a continuously differentiable function of x_0.

Let the associated cost functional, to be minimized, be defined by:

(2) $c[x(T)]$, $c[\cdot] : R^d \to R$ differentiable. We remark that (a), (b) need only hold in a "tube" of an appropriate sort about an optimal solution $(u^*(\cdot),\ x^*(\cdot))$ of (1), (2).

The Pontrjagin principle states that if $(u^*(\cdot),\ x^*(\cdot))$ is an optimal solution of (1) (2), then there exists an absolutely continuous $p(\cdot) : [0,T] \to R^d$ such that

(3) $\min_{v \in U} p(t) \cdot f(t, x^*(t), v) = p(t) \cdot f(t, x^*(t),\ u^*(t))$ in $[0,T]$. In fact $p(\cdot)$ is a solution of the adjoint equation to the linearization of (1) about $x^*(\cdot)$.

We will show how this principle follows naturally from the use of ideas from the theory of deterministic flows.

If an initial instant $s \in [0,T]$, an initial value $x \in R^n$ and a control $u(\cdot)$ are given, we write the solution of (1) satisfying $x(s) = x$ as $\xi^u_{s,t}(x)$. Here the superscript u indicates the dependence on the choice of control $u(\cdot)$; in addition we write u_t for $u(t)$. If $(u^*(\cdot),\ x^*(\cdot))$ is optimal, we write this pair as $(u^*_t,\ \xi^*_{0,t}(x_0))$ and

$$\xi^*_{0,t}(x_0) = x_0 + \int_0^t f(r, \xi^*_{0,r}(x_0), u^*_r)dr.$$

For any $x \in R^d$ and any $s \in [0,T]$ we now define $\xi^*_{s,t}(x)$ by the integral equation:

$$\xi^*_{s,t}(x) = x + \int_s^t f(r, \xi^*_{r,s}(x), u^*_r)dr. \tag{4}$$

Notice that $\xi^*_{s,t}(x)$ solves (1) as a function of t, and takes on the value x at $t = s$, but it is not necessarily optimal, (unless $s = 0,\ x = x_0$). Our assumptions imply that $\xi^*_{s,t}(x)$ is a continuously differentiable function of x. Differentiating (4) with respect to x, we obtain the integral equation defining the matrix $D_{s,t}(x) = \frac{\partial}{\partial x}(\xi^*_{s,t}(x))$:

$$D_{s,t}(x) = I + \int_s^t f_x(r, \xi^*_{r,s}(x), u^*_r)D_{s,r}(x)dr. \tag{5}$$

We now define $V_{s,t}(x)$ for any $x \in R^d$ and any s,t in $[0,T]$ by the linear integral equation:

$$(6) \qquad V_{s,t}(x) = I - \int_s^t V_{s,r}(x) f(r, \xi^*_{s,r}(x), u^*_r) dr.$$

LEMMA 1. *(from [3]).* $V_{s,t}(x) D_{s,t}(x) = I$ *for all* s,t *in* $[0,T]$, *and all* $x \in R^d$.

PROOF: Using (5) and (6), we see that

(i) $V_{s,s}(x) D_{s,s}(x) = I$

(ii) $\frac{d}{dt}\left(V_{s,t}(x) D_{s,t}(x)\right) = \left(\frac{\partial V}{\partial t}\right) D + V\left(\frac{\partial D}{\partial t}\right) =$

$= -V_{s,t}(x) f(t, \xi^*_{s,t}(x), u^*_t) D_{s,t}(x) + V_{s,t}(x) f(t, \xi^*_{s,t}(x), u^*_t) D_{s,t}(x)$

$= 0.$

Thus, $V_{s,t}(x) = [D_{s,t}(x)]^{-1}$ for all s,t in $[0,T]$ and all $x \in R^d$. In particular, we conclude that $D_{s,t}(x)$ is always invertible.

Next, for any continuous map (path) $z_t : [0,T] \to R^d$, we consider the composite map $\varphi : t \mapsto \xi^*_{s,t}(z_t)$, i.e., the function $\phi(t) \equiv \xi_{s,t}(z_t)$ defined for s and t in $[0,T]$ by the integral equation:

$$(7) \qquad \phi(t) = z_s + \int_s^t f(r, \phi(r),\ u(r)) dr + \int_s^t D_{s,r}\left(\frac{dz_r}{dr}\right) dr.$$

This equation is obtained from (4) by differentiation with respect to t.

We now perturb the given optimal control $u^*(\cdot)$ in the by now standard manner:

$$(8) \qquad u_t = \begin{cases} u^*_t & \text{outside} \quad [s, s+h], \\ \tilde{u} \in V & \text{inside} \quad [s, s+h], \end{cases}$$

and define the curve $z_t : [0,T] \to R^d$ by the somewhat improbable integral equation (note that the subscript (s,r) is reversed from (4)):

$$(9) \qquad z_t = x + \int_s^t [D_{s,r}(z_r)]^{-1} \left[f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r)\right] dr.$$

LEMMA 2.

$$\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x) \quad \textit{for all} \quad x \in R^d, \ s \textit{ and } t \textit{ in } [0,T].$$

PROOF: By (7) and (9),

$$\xi^*_{s,t}(z_t) = x + \int_s^t f(r, \xi^*_{s,t}(z_r), u^*_r)dr +$$

$$+ \int_s^t D_{s,r}(z_r)\left[D_{s,r}(z_r)\right]^{-1}\left[f(r, \xi^*_{s,t}(z_r), u_r) - f(r, \xi^*_{s,t}(z_r), u^*_r)\right]dr$$

$$= x + \int_s^t f(r, \xi^*_{s,t}(z_r), u_r)dr.$$

The assertion follows from the uniqueness of solutions to (1).

3. The Minimum Principle. If we define $x = \xi^*_{0,s}(x_0)$, for a given $s \in [0,T]$, then the optimal cost can be written

$$c[x^*_T] = c\left[\xi^*_{0,T}(x_0)\right] = c\left[\xi^*_{s+h,T}\left(\xi^*_{s,s+h}(x)\right)\right],$$

for any $h \geq 0$, $s \in [0,T]$, $s + h \in [0,T]$, with $x = \xi^*_{0,s}(x_0)$.

Since $u(\cdot)$ cannot give a lower cost than $u^*(\cdot)$,

$$c\left[\xi^*_{s+h,T}\left(\xi^*_{s,s+h}(x)\right)\right] \leq c\left[\xi^*_{s+h,T}\left(\xi^u_{s,s+h}(x)\right)\right] \leq$$

$$\leq c\left[\xi^*_{s+h,T}\left(\xi^u_{s,s+h}(z_{s+h})\right)\right],$$

thus

$$c\left[\xi^*_{s,T}(x)\right] - c\left[\xi^*_{s,T}(z_{s+h})\right] \leq 0 \tag{10}$$

for all $s \in [0,T]$, $h \geq 0$, $(s+h) \in [0,T]$.

This, with judicious application of Mean Value Theorems and letting h tend to zero, leads to the Pontrjagin principle. In fact, by the Mean-Value Theorem,

$$\left[\frac{\partial c}{\partial \xi}\right] \cdot \left[\left(\xi^*_{s,T}(x) - \xi^*_{s,T}(z_{s+h})\right] \leq 0,\right.$$

for s, h as above, where the gradient of $c(\cdot)$ is evaluated somewhere on the line between $\xi^*_{s,T}(x)$ and $\xi^*_{s,T}(z_{s+h})$ in R^d. Since we will shortly let h decrease to zero, this evaluation point will become $\xi^*_{s,T}(x) = x^*_T$.

It follows from (4) and our assumptions on f that

(11) $$[\frac{\partial c}{\partial \xi}] D_{s,T} [x - z_{s+h}] \leq 0,$$

but here the rows of $D_{s,T} = \partial \xi_{s,T}(x)/\partial x$ are evaluated at perhaps different points between x and z_{s+h}, because the Mean-Value Theorem is only valid for real-valued mappings and hence must be applied to each component of $\xi^*_{s,T}$. From (9) we can write

(12) $$x - z_{s+h} = \int_s^{s+h} [D_{s,r}(z_r)]^{-1} [f(r, \xi^*_{s,r}(z_r), u^*_r) - f(r, \xi^*_{s,r}(z_r), u_r] dr.$$

Combining (11) and (12), dividing by $h > 0$ and letting h go to zero, we obtain (noting that $D_{ss} = I,\ z_s = x = \xi^*_{s,s}$):

$$0 \geq [\frac{\partial c}{\partial \xi}(x^*_T)] D_{s,T}(\xi^*_{s,s}) \cdot [f(s, \xi^*_{s,s}, u^*_s) - f(s, \xi^*_{s,s}, \tilde{u})].$$

This is the Pontrjagin principle (3) with

(13) $$p(s) = [\frac{\partial c}{\partial \xi}(x^*_T)] \frac{\partial \xi^*_{s,T}}{\partial x}(\xi^*_{s,s}).$$

4. The adjoint equation.

By the semigroup property of the solution flows, for $0 \leq s \leq t \leq T$;

(14) $$\xi^*_{0,t}(x_0) = \xi^*_{s,t}(\xi^*_{0,s}(x_0)).$$

Writing $D^*_{s,t} = \frac{\partial \xi^*_{s,t}}{\partial x}(x)$ and differentiating (14) by the chain rule

(15) $$D^*_{0,t} = D^*_{s,t}(\xi^*_{0,s}(x_0)) D^*_{0,s}(x_0).$$

From (13)

$$p(s) = \frac{\partial c}{\partial \xi}(x^*_T) D^*_{s,T}(\xi^*_{s,s}).$$

Therefore, using (15)

$$p(s)D^*_{0,s}(x_0) = \frac{\partial c}{\partial \xi}(x^*_T)D^*_{0,T}(x_0) = \text{ constant.} \tag{16}$$

Differentiating (16) in s

$$pdD^* + (dp)D^* = 0.$$

That is $dp = (-pdD^*)D^{*-1}$. (We have noted in Lemma 1 that $D^{*-1}_{0,s} = V^*_{0,s}$ exists.)

From (5)

$$dD^*_{0,s} = f_x(s,\ \xi^*_{0,s}(x_0),\ u^*_s)D^*_{0,s}\,ds.$$

Therefore, $p(s)$ is the solution of the equation

$$dp(s) = -p(s)f_x(s,\ \xi^*_{0,s}(x_0),\ u^*_s)ds$$

with initial condition

$$p(0) = \frac{\partial c}{\partial \xi}(x^*_T)D^*_{0,T}.$$

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Integration by parts and the Malliavin calculus

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1. Introduction.

From a very simple representation of the integrand in the integral representation of a martingale, we derive an integration by parts formula. This is used to give a new proof of the existence of a density of a diffusion process under the hypothesis that the inverse of the Malliavin matrix is in some L^p-space, a result implied by Hörmander's condition H1.

Following Malliavin's original proof of this result there have been other approaches to what is now known as Malliavin's calculus, including those of Stroock [17], Shigekawa [16], Bismut [4], Bichteler and Fonken [2], and Norris [15]. The main simplification in this paper is the observation that no infinite dimensional calculus of variations is required. This calculus can be replaced by ordinary differentiation in finite dimensional spaces.

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2. Some history and the H1 condition.

Let us consider the unique solution $\xi_{s,t}(x)$ of the stochastic differential equation

$$d\xi_{s,t}(x) = X_0(t, \xi_{s,t}(x))dt + X_i(t, \xi_{s,t}(x))dw_t^i$$

$$\xi_{s,s}(x) = x \in \mathbb{R}^d \tag{1}$$

where $(w_t) = (w_t^1, \ldots, w_t^m)$ is an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $X_0, X_1, \ldots, X_m$ are smooth vector fields on $[0,\infty) \times \mathbb{R}^d$, all of whose derivatives are bounded.

It is a well known fact from harmonic analysis that $\xi_{0,T}(x)$ has a density if

$$|Ec_\xi(\xi_{0,T}(x_0))| \leq K \sup_{x \in \mathbb{R}^d} |c(x)|, \tag{2}$$

where c is any bounded, smooth function with bounded derivatives [14,17,4,20]. Using different methods Malliavin [14], Stroock [17], Shigekawa [16], and Bismut [4] showed that (2) is true if the inverse of the Malliavin matrix M_{0T} is in some $L^p(\Omega)$, and they linked this result with Hörmander's famous result to conclude that M_{0T}^{-1} is in all $L^p(\Omega)$, $p < \infty$, if Hörmander's condition H1 is satisfied:

Condition H1: $X_1, \ldots, X_m$, $[X_i, X_j]$, $[X_i, [X_j, X_k]], \ldots,$ $i, j, k = 0, \ldots, m$ at x_0 span $\mathbb{R}^d$.

Malliavin's approach is based on a function space martingale calculus which comes from the Ornstein–Uhlenbeck process on Wiener space [14]; this is now known as Malliavin's calculus of variations. Shigekawa [16] provided an alternative formulation which relies on a Sobolev-type extension of Fréchet derivatives with Wiener measure replacing the Lebesgue measure in the finite dimensional situation, and he makes no use at all of the Ornstein–Uhlenbeck process. Stroock [17,18] also avoids this process in his entirely functional-analytic reformulation of the Malliavin calculus. So far the approaches of Shigekawa and Stroock (also cf. Ikeda and Watanabe's contribution [12]) are reformulations of Malliavin's approach.

Roughly speaking, these approaches rely on the analysis of a differential operator $\mathcal{L}$, which may be seen on the one hand as an operation on the Wiener chaos decomposition of a Brownian functional $F(w)$

$$\mathcal{L}F(w) = \sum_{m=1}^{\infty} m \int_0^T \cdots \int_0^{t_2} f_m(t_1, \ldots, t_m) dw_{t_1} \ldots dw_{t_m},$$

or as the generator of a time changed Brownian sheet $\{S_\tau(t) \mid (\tau, t) \in [0, \infty)^2\}$, namely

$$V_\tau(t) = e^{-\frac{1}{2}\tau} S_{e^\tau}(t)$$

seen as a process on $C(0, \infty)$. For a "good" function c, we then find

$$c'(F) = \frac{1}{2}\big(- \mathcal{L}(Fc(F)) + c(F)\mathcal{L}F + F\mathcal{L}F\big) \cdot A^{-1}, \tag{3}$$

where A is the inverse of the Malliavin matrix $A = (DF, DF) = \sum_i (D_{h^i}F)^2$ and D_{h^i} is the directional derivative in the direction of the integrated element h^i of a complete orthonormal system on $[0, T]$. The analysis of the right hand side of (3) then leads to a bound on $E|c'(F)|$ as required in (2).

Zakai pointed out that $\mathcal{L}F$ may also be seen as the L^2-limit of

$$\frac{F(w) - E[F(\sqrt{1-\varepsilon}\ w + \sqrt{\varepsilon}\tilde{w}) \mid \mathcal{F}^w]}{\varepsilon},$$

where the relation to the generator of the infinite dimensional Ornstein–Uhlenbeck process becomes apparent [19,20], as this non-coherent derivative may be interpreted as

$$\begin{aligned} &\frac{\partial F(\zeta w)}{\partial \zeta}\Big|_{\zeta=1} - \sum \frac{\partial^2}{\partial \varepsilon^2} F(w + \varepsilon \int h^i_s ds)\Big|_{\varepsilon=0} \\ &= D^\zeta F - \text{trace } D^2 F. \end{aligned} \tag{4}$$

Bismut however gives a different approach which expresses the Wiener space derivatives as function space derivatives in a Girsanov functional. The basic idea here is a

perturbation of Brownian motion by a small drift $\varepsilon \cdot \int u_s ds$ (u_s a predictable function). Then

$$D_u F(w) = \frac{\partial}{\partial \varepsilon} F(w + \varepsilon \int u ds)\Big|_{\varepsilon=0}.$$

However,

$$E[F(w)] = E[F(w + \varepsilon \int u ds) \cdot \gamma_T]$$

where γ_T is the Girsanov functional, the solution of

$$\gamma_T = 1 - \varepsilon \int \gamma_s u_s dw_s.$$

With

$$\begin{aligned} E[F(w)] &= E[F(w + \int \varepsilon u d_s) \cdot \gamma_T] \\ &\approx E[F(w)] + \varepsilon E[D_u F(w)] - \varepsilon E[F(w) \int u ds] \end{aligned}$$

we find the Bismut integration by parts formula

$$E[F(w) \int u_s dw_s] = E[D_u F(w)].$$

Applying this to "nice" functions $c(F) \cdot g$, formally we find

$$E[c(F)(D_u F)^{-1} \int u dw] = E\big[c'(F)(D_u F)^{-1} D_u F + c(F) D_u\big((D_u F)^{-1}\big)\big]$$

and

$$Ec'(F) = E\big(c(F)(D_u F)^{-1} \int u dw\big) - E\big(c(F) D_u (D_u F)^{-1}\big). \tag{5}$$

$(D_u F)$ now plays the role of the Malliavin matrix, and the assumption that $D_u F > 0$ for a suitable predictable (u_s) leads to a bound on $|Ec'(F)|$ in (4) as required in (2).

In the survey article, [20], Zakai points out that the Malliavin and Bismut approaches are not equivalent.

We follow here, more-or-less, the Bismut approach, but where Bismut considers variations in a function space our formulation reduces the Malliavin calculus to differentiation

in a finite dimensional space for the situation where the Wiener functional is just a solution of a diffusion equation as in (1), $F(w) = \xi_{0T}(x_0)$. The key observation which leads to our result is a martingale representation formula which might be seen as coming from the folklore of mathematics, but it provides us with a new formulation of the integration by parts formula, which – as is well known – always plays the fundamental role in Malliavin's calculus.

3. Representation of martingales.

Consider the solution $\xi_{0,t}(x_0)$ of (1) and let c be a twice continuously differentiable function for which $c(\xi_{0,T}(x_0))$ and the components of $c_\xi(\xi_{0,T}(x_0))$ are integrable. We then have the following representation for the right continuous version of $E[c(\xi_{0,T}(x_0)) \mid \mathcal{F}_t] =: m_t$.

THEOREM 3.1. *The martingale m_t, $0 \le t \le T$, has a representation as*

$$m_t = E[c(\xi_{0,T}(x_0))] + \int_0^t \gamma_i(s) dw_s^i$$

with

$$\gamma_i(s) = E[c(\xi_{0,T}(x_0)) D_{0,T} \mid \mathcal{F}_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)), \tag{6}$$

where $D_{s,t}$ is the Jacobian of the stochastic flow,

$$D_{s,t} = \frac{\partial \xi_{s,t}}{\partial x}.$$

Note that from the following theorem cited from [3,8] $D_{s,t}$ exists as a solution of a stochastic differential equation.

THEOREM 3.2. *There is a map $\xi : \Omega \times [0,\infty) \times [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that*

(i) for $0 \le s \le t \le T$, $x \in \mathbb{R}^d$, $\xi_{s,t}(x)$ is the essentially unique solution of (1);

(ii) for each ω, s, t the map $\xi_{s,t}(\omega, \cdot)$ is $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with a Jacobian which satisfies

$$dD_{s,t} = \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}dt + \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}dw_t^i$$

$$D_{s,s} = I, \quad \textit{the identity matrix;}$$

(iii) the second derivative $W_{s,t} = \dfrac{\partial^2 \xi_{s,t}}{\partial x^2}$ satisfies

$$\begin{aligned} dW_{s,t} &= \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))W_{s,t}dt + \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))W_{s,t}dw_t^i \\ &+ \frac{\partial^2 X_0}{\partial \xi^2}(t, \xi_{s,t}(x))D_{s,t} \otimes D_{s,t}dt + \frac{\partial^2 X_i}{\partial \xi^2}(t, \xi_{s,t}(x))D_{s,t} \otimes D_{s,t}dw_t^i \\ W_{s,s} &= 0 \in \mathbb{R}^d \otimes \mathbb{R}^d \otimes \mathbb{R}^d. \end{aligned}$$

Proof of 3.1: Any $\mathcal{F}_t$-martingale (m_t) may be represented as

$$m_t = m_0 + \int_0^t \gamma_i(s)dw_s^i$$

for a predictable integrand γ_i. As $\xi_{0,t}(x_0)$ is Markov

$$\begin{aligned} m_t &= E[c(\xi_{0,T}(x_0)) \mid F_t] \\ &= E[c(\xi_{t,T}(x)) \mid F_t] \\ &= E_{t,x}[c(\xi_{t,T}(x))] \\ &=: V(t,x), \qquad \text{where} \quad x = \xi_{0,t}(x_0). \end{aligned}$$

Then applying Itô's rule to $V(t,x)$, $x = \xi_{0,t}(x_0)$ gives

$$\begin{aligned} V(t, \xi_{0,t}(x_0)) &= V(0, x_0) + \int_0^t \Big(\frac{\partial V}{\partial s} + LV\Big)ds \\ &+ \int_0^t \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0))X_i(s, \xi_{0,s}(x_0))dw_s^i \end{aligned}$$

with

$$L = X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum X_k^i X_k^j \frac{\partial^2}{\partial x_i \partial x_j}.$$

As (m_t) is a martingale, from the uniqueness of semimartingale decomposition we must have

$$\left(\frac{\partial V}{\partial s} + LV\right) = 0$$

and

$$\gamma_i(s) = \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0)) X_i(s, \xi_{0,s}(x_0)).$$

Differentiating V we thus arrive at

$$\gamma_i(s) = E[c_\xi(\xi_{0,T}(x_0)) D_{0,T} \mid F_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)). \tag{7}$$

□

Now let $u(s) = (u_1(s), \ldots, u_m(s))$ be a square integrable predictable process. Applying the above representation we find the desired integration by parts formula.

THEOREM 3.3. *Under the above assumptions the following equality holds*

$$E\Big[c(\xi_{0,T}(x_0) \int_0^T u_i(s) dw_s^i\Big] = E \int_0^T E[c_\xi(\xi_{0,T}(x_0)) D_{0,T} \mid F_s] D_{0,s}^{-1} X_i(s) u_i(s) ds$$

$$= E\Big[c_\xi(\xi_{0,T}(x_0)) D_{0,T} \int_0^T D_{0,s}^{-1} x_i(s) u_i(s) ds\Big]$$

by Fubini's theorem.

In particular, putting $u_i(s) = (D_{0,s}^{-1} X_i(s))^*$ and considering the product function $h(\xi_{0,T}(x_0)) = c(\xi_{0,T}(x_0)) g(\xi_{0,T}(x_0)$ we have

THEOREM 3.4.

$$E[c(\xi_{0,T}(x_0)) g(\xi_{0,T}(x_0))] \int_0^T (D_{0,s}^{-1} x_i(s))^* dw_s^i = E[(c_\xi g + c g_\xi) D_{0T} M_{0T}],$$

where

$$M_{s,t} = \sum_{i=1}^m \int_s^t D_{s,u}^{-1} X_i(u) X_i^*(u) D_{s,u}^{*-1} du.$$

$M_{s,t}$ is the Malliavin matrix. □

In order to obtain a bound on c_ξ we now would like to take

$$g = M_{0,T}^{-1} D_{0,T}^{-1},$$

but this function not only depends on $\xi_{0,T}$. To get around this difficulty we have to enlarge the system in the following way.

4. Existence of a density for $\xi_{0,T}(x_0)$.

When enlarging the system the results of 3.2 might no longer hold for $\xi_{s,t}$ replaced by the new system as the coefficients are no longer bounded.

We consider the flow defined by (1), its Jacobian $D_{s,t}$, the martingale $R_{s,t}(x) = \int_s^t (D_{s,u}^{-1} X_i(u))^* dw_u^i$, and the inverse of the Malliavin matrix $M_{s,t} = \sum \int_s^t D_{s,u}^{-1} X_i(u) X_i^*(u) D_{s,u}^{*-1} du$. Then for

$$\phi^{(0)}(w,s,t,x) = \xi_{s,t}(x), \qquad x = \xi_{0,s}(x_0)$$

$$D_{s,t}^{(0)}(x) = D_{s,t}(x), \qquad D = D_{0,s}(x_0)$$

$$D_{0,t}^{(0)}(x_0) = D_{s,t} D$$

$$R_{s,t}^{(0)}(x) = \int_s^t (D_{s,u}^{-1} X_i(u))^* dw'u \tag{8}$$

$$R_{0,t}^{(0)} = R + D^{-1} R_{s,t}^0(x), \qquad R = R_{0,s}^0$$

$$M_{s,t}^{(0)} = M_{s,t}(x)$$

$$M_{0,t}^{(0)} = M + D^{-1} M_{s,t}(x) D^{*-1}, \qquad M = M_{0,s}^0$$

the enlarged system $\phi^{(1)} = (\phi^0, D^0, R^0, M^0)$ is Markov. We now would like to apply a result similar to 3.2 to this enlarged system. Introduce the set $S_\alpha(d_1,\ldots,d_k)$, $\alpha, d, d_1, \ldots, d_k$

positive integers, of C^∞ functions $X : \mathbb{R}^d \to \mathbb{R}^d$ of the triangular form

$$X(x) = \begin{pmatrix} X^{(1)}(x^1) \\ X^{(2)}(x^1, x^2) \\ \vdots \\ X^k(x^1, x^2, \dots, x^k) \end{pmatrix} \qquad \text{for} \qquad x = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}, \qquad x^i \in \mathbb{R}^{d_i}$$

and $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$, which satisfy

$$\|X\|_{S(\alpha,N)} = \sup_{x \in \mathbb{R}^d} \Big(\sup_{0 \le n \le N} \frac{|DX(x)|}{1 + |x|^\alpha} \vee \sup_{1 \le j \le k} |D_j X^{(j)}(x)| \Big) < \infty$$

for all N.

Note that ϕ^1 is Markov with coefficients in $S(d,\ d+d^2,\ 2d+d^2,\ 2d+d^2)$, and following Norris [15] we may state the extension of 3.2.

THEOREM 4.1. *Let $x_0, x_1, \dots, x_m \in S_\alpha(d_1, \dots, d_k)$. Then there is a map $\phi : \Omega \times [0,\infty)^2 \times \mathbb{R}^d \to \mathbb{R}^d$ such that*

(i) for $0 \le s \le t$, $x \in \mathbb{R}^d$, ϕ is the essentially unique solution of

$$dx_t = X_0(x)dt + X_i(x)dw_t^i, \quad x_s = x;$$

(ii) for all (ω, s, t), ϕ is C^∞ with derivatives of all orders satisfying stochastic differential equations;

(iii)

$$\sup_{|x| \le R} E\Big[\sup_{s \le u \le t} |D^n \phi(\omega, s, u, x)|^p \Big]$$
$$\le C(p, R, N, d_1, \dots, d_k,\ \alpha,\ \|X_0\|_{S_{\alpha,N}}, \dots, \|X_m\|_{S_{\alpha,N}}).$$

□

Furthermore, we can consider the Jacobian of $\phi^{(1)}$, say $D^{(1)}$, and construct $R_{s,t}^{(1)} = \int_s^t D_{s,u}^{(1)-1} X_i^{(1)}(u) dw_u^i$, and let $M_{s,t}^{(1)} = \langle R_{s,t}^{(1)} \otimes R_{s,t}^{(0)*} \rangle$ be the predictable quadratic variation of R^1 and R^{0*}.

This 4-tuple defines $\phi^{(2)} = (\phi^{(1)}, D^{(1)}, R^{(1)}, M^{(1)})$ and inductively we can proceed to define $\phi^{(n)}$ for all n, and Norris' result holds for all $\phi^{(n)}$.

Now apply 3.4 to $c(\phi^{(0)}) \cdot g(\phi^{(1)})$ to obtain:

COROLLARY 4.2.

$$E[c(\phi^{(0)})g(\phi^{(1)}) \otimes R^{(0)}] = E[(\nabla_0 c)(\phi^0)g(\phi^1)D_{0,T}M_{0,T}]$$

$$+ E[c(\phi^0)(\nabla_1 g)(\phi_1)D^1 M^1],$$

and for

$$g(\phi^{(1)}) = M_{0,T}^{-1} D_{0,T}^{-1}$$

we find

$$E[c_\xi(\xi_{0,T}(x_0))] = E[c(\xi_{0,T}(x_0))M_{0,T}^{-1}D_{0,T}^{-1} \otimes R_{0,T}]$$

$$- E[c(\xi_{0,T}(x_0))(\nabla_1 g)D_{0,T}M_{0,T}D_{0,T}^{(1)}M_{0,T}^{(1)}].$$

□

An application of Jensen's, Burkholder's, and Gronwall's inequalities with Norris' result implies that all terms, except possibly $M_{0,T}^{-1}$, are in all L^p, $p < \infty$. If now we assume that $M_{0,T}^{-1}$ is in some L^p, e.g., if we assume H1 to hold, then we have the desired result.

THEOREM 4.3. *Let $\xi_{0,T}(x_0)$ be the solution of (1) and c a bounded C^∞ function with bounded derivatives. Then if $M_{0,T}^{-1}$ is in some L^p*

$$|E[c_\xi(\xi_{0,T}(x_0)]| \leq K \sup_{x \in \mathbb{R}^d} |c(x)|.$$

With this result, D. Williams' 'ridiculous' example on the existence of a density for the Brownian motion really becomes trivial:

$$|E[c'(w_1)]| \leq |E[c(w_1) \cdot w_1]| \leq \sup_{x \in \mathbb{R}^d} c(x) \cdot \text{const}.$$

5. Application.

The Malliavin calculus could not have attracted so much attention if there were not many important applications, together with the remarkable fact that it links the Hörmander partial differential equation methods with probabilistic aspects. Within stochastic analysis it provides many helpful tools, such as, for example, the integration by parts formula which is equivalent to a martingale representation theorem. In filtering theory, D. Michel and J.M. Bismut [5] used the calculus of variations to prove the existence of densities for optimal filters, and Jacod and Bichteler [1] extended these results to diffusion processes with jumps.

Many of these results can be simplified by using the finite dimensional calculus developed above. The full details are found in [9,10,11].

J.M. Bismut [7] (also cf. [13]) applied the results from Malliavin's calculus to the theory of index theorems in algebraic topology and to large deviations problems [6].

Recently, there have been several attempts to develop a notion of anticipative stochastic integrals. This would allow one to consider functions $u(s)$ above which might not be predictable and, in turn, this would then allow the development of Bismut's Malliavin calculus to its full strength.

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Martingale Representation and Hedging Policies

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The integrand, when a martingale under an equivalent measure is represented as a stochastic integral, is determined by elementary methods in the Markov situation. Applications to hedging portfolios in finance are described.

martingale representation * Girsanov's theorem * stochastic flow * diffusion * hedging portfolio

1. Introduction.

In the modern theory of option pricing and hedging, the representation of martingales as stochastic integrals plays a central role. Since the corresponding integrands immediately lead to hedging strategies, it is of considerable interest to find explicit expressions for these integrands.

The martingale representation result and its background is fully described in the paper of Ocone [12], where the problem is discussed using methods of the Malliavin calculus and weak differentiability in certain Sobolev spaces. In a recent paper by Ocone and Karatzas [13] the representation result of [12] is applied to determine optimal portfolios and hedging strategies.

In the Markov case elementary methods, which do not use the Malliavin calculus in function space, are employed by Elliott and Kohlmann in [5] and [6] to determine the integrands in certain stochastic integrals. Indeed, all that is used is the Markov property and the Itô differentiation rule.

The present paper extends the representation result of [6] to the situation where the martingale representation takes place with respect to an equivalent measure whose Girsanov exponential is defined in terms of a Markov integrand. The motivation for the Girsanov measure transformation is developed by Harrison, Kreps and Pliska [8], [9]. A Markov Girsanov transform clearly introduces a new integrand in the martingale representation, and this is made explicit in Theorem 3.1. It is possible this result could be derived as a corollary of the general result of Haussman and Ocone, see [12]. However any such

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relation is certainly not transparent and our proof, again in the Markov case, is simple and direct.

The application of our martingale representation result to option pricing is described in Section 4. Stock price dynamics which give explicit, closed form expressions for hedging policies appear hard to find. However, in Section 5 we show how our result gives the hedging policy in the well known Black–Scholes case, [1], of log-normal prices with constant drift and variance.

2. Dynamics.

Suppose $w = (w^1, \ldots, w^m)$ is an m-dimensional Brownian motion defined for $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, P)$. Consider the d-dimensional stochastic differential equation

$$dx_t = f(t, x_t)dt + \sigma(t, x_t)dw_t \tag{2.1}$$

for $t \geq 0$, where $f : [0, \infty) \times R^d \to R^d$ and $\sigma : [0, \infty) \times R^d \to R^d \otimes R^m$ are measurable functions which are three times differentiable in x, and which, together with their derivatives, have linear growth in x. Write $\xi_{s,t}(x)$ for the solution of (2.1) for $t \geq s$ having initial condition $\xi_{s,s}(x) = x$. Then from the results of Bismut [2] or Kunita [11] there is a set $N \subset \Omega$ of measure zero such that for $w \notin N$ there is a version of $\xi_{s,t}(x)$ which is twice differentiable in x and continuous in t and s.

Write $D_{s,t}(x) = \dfrac{\partial \xi_{s,t}(x)}{\partial x}$ for the Jacobian of the map $x \to \xi_{s,t}(x)$; then it is known that D is the solution of the linearized equation

$$dD_{s,t}(x) = f_x(t, x_t)D_{s,t}(x)dt + \sigma_x(t, x_t)D_{s,t}(x)dw_t$$

with initial condition $D_{s,s}(x) = I$, the $d \times d$ identity matrix. The inverse $D_{s,t}^{-1}(x)$ exists; see [2].

Suppose $g : [0, \infty) \times R^d \to R^m$ satisfies similar conditions to those of f and define the (scalar) exponential $M_{s,t}(x)$ by

$$\begin{aligned} M_{s,t}(x) &= 1 + \int_s^t M_{s,r}(x) g(r, \xi_{s,r}(x)) \cdot dw_r \\ &= 1 + \int_s^t dw_r^* \cdot g(r, \xi_{s,r}(x)) M_{s,r}(x), \end{aligned} \tag{2.3}$$

where * denotes adjoint and $\cdot$ inner product in R^d.

Write $\{F_t\}$ for the right continuous complete family of σ-fields generated by w. If, for example, g further satisfies a linear growth condition

$$|g(t, x)| \leq K(1 + |x|)$$

a new probability measure $\widetilde{P}$ can be defined by putting

$$\left.\frac{d\widetilde{P}}{dP}\right|_{F_t} = M_{0,t}(x_0).$$

Girsanov's theorem then implies that $\widetilde{w}$ is an $\{F_t\}$ Brownian motion under $\widetilde{P}$ where

$$d\widetilde{w}_t = dw_t - g(t, \xi_{0,t}(x_0))dt. \tag{2.4}$$

Let $c : R^d \to R$ be a C^2 function which, together with its derivatives, has linear growth, and for $0 \le t \le T$ consider the $\widetilde{P}$ martingale

$$N_t = \widetilde{E}[c(\xi_{0,T}(x_0)) \mid F_t].$$

Then from, for example, Theorem 16.22 of [4] N_t has a representation for $0 \le t \le T$ as

$$N_t = N_0 + \int_0^t \gamma_s d\widetilde{w}_s, \tag{2.5}$$

where γ is an $\{F_t\}$ predictable process such that

$$\int_0^T \widetilde{E}|\gamma_s|^2 ds < \infty.$$

3. Martingale Representation.

THEOREM 3.1.

$$\gamma_t = \widetilde{E}\Big[\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r, \xi_{0,r}(x_0))D_{0,r}(x_0) \cdot c(\xi_{0,T}(x_0))$$
$$+ c_\xi(\xi_{0,T}(x_0))D_{0,T}(x_0) \mid F_t\Big] D_{0,t}^{-1}(x_0) \cdot \sigma(t, \xi_{0,t}(x_0)).$$

PROOF. For $0 \le t \le T$ write $x = \xi_{0,t}(x_0)$. By the semigroup property of stochastic flows, which follows from the uniqueness of solutions of (2.1),

$$\xi_{0,T}(x_0) = \xi_{t,T}(\xi_{0,t}(x_0)) = \xi_{t,T}(x). \tag{3.1}$$

Differentiating (3.1) we see

$$D_{0,T}(x_0) = D_{t,T}(x)D_{0,t}(x_0). \tag{3.2}$$

Furthermore,

$$M_{0,T}(x_0) = M_{0,t}(x_0)M_{t,T}(x). \tag{3.3}$$

For $y \in R^d$ define $\widetilde{V}(t, y) = E[M_{t,T}(y)c(\xi_{t,T}(y))]$, and consider the martingale

$$N_t = \widetilde{E}[c(\xi_{0,T}(x_0)) \mid F_t]$$
$$= \frac{E[M_{0,T}(x_0)c(\xi_{0,T}(x_0)) \mid F_t]}{E[M_{0,T}(x_0) \mid F_t]}$$

$$= E[M_{t,T}(x)c(\xi_{t,T}(x)) \mid F_t]$$
$$= E[M_{t,T}(x)c(\xi_{t,T}(x))], \quad \text{by the Markov property.}$$

Then from Lemma 14.18 of [4]

$$N_t = \widetilde{V}(t,x).$$

We noted above that $\xi_{t,T}(x)$ is twice differentiable in x; the differentiability of $E[M_{t,T}(x)c(\xi_{t,T}(x))]$ in t can be established by writing the backward equation for $(M_{t,T}(x), \xi_{t,T}(x))$ as in [11].

Under $\widetilde{P}$, $\xi_{0,t}(x_0)$ is given by the equation

$$\xi_{0,t}(x_0) = x_0 + \int_0^t \left(f(s,\xi_{0,s}(x_0)) + \sigma \cdot g(s,\xi_{0,s}(x_0))\right)ds$$
$$+ \int_0^t \sigma(s,\xi_{0,s}(x_0))d\widetilde{w}_s. \tag{3.4}$$

If $\widetilde{V}(t,\xi_{0,t}(x_0))$ is expanded by the Itô rule we see

$$\widetilde{V}(t,x) = \widetilde{V}(t,\xi_{0,t}(x_0)) = N_t$$
$$= \widetilde{V}(0,x_0) + \int_0^t \left(\frac{\partial \widetilde{V}}{\partial t}(s,\xi_{0,s}(x_0)) + L\widetilde{V}(s,\xi_{0,s}(x_0))\right)ds$$
$$+ \int_0^t \frac{\partial \widetilde{V}}{\partial x}(s,\xi_{0,s}(x_0))\sigma(s,\xi_{0,s}(x_0))d\widetilde{w}_s. \tag{3.5}$$

Here

$$L = \sum_{i=1}^d \left(f^i + \sum_{j=1}^m \sigma_{ij}g^j\right)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}\,\frac{\partial^2}{\partial x_i \partial x_j}$$

where $a(t,x_i) = (a_{ij}(t,x_t))$ is the matrix $\sigma\sigma^*$. Now N_t is a special semimartingale, so the decompositions (2.5) and (3.5) must be the same. As there is no bounded variation term in (2.5) we see immediately, similarly to [7], that

$$\frac{\partial \widetilde{V}}{\partial t}(s,\xi_{0,s}(x_0)) + L\widetilde{V}(s,\xi_{0,s}(x_0)) = 0$$

with $\widetilde{V}(T,x) = c(x)$. Also

$$\gamma_s = \frac{\partial \widetilde{V}}{\partial x}(s,\xi_{0,s}(x_0))\sigma(s,\xi_{0,s}(x_0)).$$

However, $\xi_{t,T}(x) = \xi_{0,T}(x_0)$ so, from the differentiability and linear growth of g:

$$\frac{\partial \widetilde{V}(t,x)}{\partial x} = E\Big[\frac{\partial M_{t,T}(x)}{\partial x} c(\xi_{0,T}(x_0)) + M_{t,T}(x)\frac{\partial c}{\partial x}(\xi_{t,T}(x))\Big].$$

Again using the existence of solutions of stochastic differential equations which are differentiable in their initial conditions, we have from (2.3)

$$\frac{\partial M_{t,T}(x)}{\partial x} = \int_t^T d\widetilde{w}_r^* \cdot g(r,\xi_{t,r}(x))\frac{\partial M_{t,r}(x)}{\partial x}dr + \int_t^T dw_r^* \cdot g_\xi(r,\xi_{t,r}(x))\frac{\partial \xi_{t,r}(x)}{\partial x}\ M_{t,r}(x). \tag{3.6}$$

However, we can solve (3.6) by variation of constants and obtain

$$\frac{\partial M_{t,T}(x)}{\partial x} = M_{t,T}(x) \cdot \int_t^T d\widetilde{w}_r^* \cdot g_\xi(r,\xi_{t,r}(x))D_{t,r}(x). \tag{3.7}$$

The result can be verified by differentiation, because (3.6) has a unique solution. Therefore, with $x = \xi_{0,t}(x_0)$,

$$\begin{aligned}\frac{\partial \widetilde{V}(t,x)}{\partial x} &= E\Big[M_{t,T}(x)\Big\{\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r,\xi_{t,r}(x))D_{t,r}(x)\cdot c(\xi_{0,T}(x_0))\\ &\qquad + c_\xi(\xi_{t,T}(x))D_{t,T}(x)\Big\}\Big]\\ &= \widetilde{E}\Big[\int_t^T d\widetilde{w}_r^* \cdot g_\xi(r,\xi_{0,r}(x_0))D_{0,r}(x_0)\cdot c(\xi_{0,T}(x_0))\\ &\qquad + c_\xi(\xi_{0,T}(x_0))D_{0,T}(x_0) \mid F_t\Big]D_{0,t}^{-1}(x_0),\end{aligned}$$

and the result follows.

REMARK 3.2. The result extends immediately to functions for which a generalized Itô formula holds; this class includes convex functions and differences of two convex functions. See Karatzas and Shreve [10].

4. Hedging Portfolios.

It is shown in Harrison and Pliska [9] that hedging policies arise from a martingale representation under an equivalent measure. Consequently, we give an application of Theorem 3.1 in this section.

Consider a vector of d stocks

$$S = (S^1, \ldots, S^d)'$$

whose prices are described by a system of stochastic differential equations of the form

$$dS_t^i = S_t^i\Big(\mu^i(t,S_t)dt + \sum_{j=1}^d \lambda_{ij}(t,S_t)dw_t^j\Big).$$

Models of this kind are usual in finance. When the μ^i and λ_{ij} are constant we have the familiar log-normal stock price. For economic reasons, so that the claim is attainable, see [9], the number of sources of noise, that is the dimension of the Brownian motion w, is taken equal to the number of stocks. $\Lambda_t = \Lambda(t, S) = (\lambda_{ij}(t, S))$ is, therefore, a $d \times d$ matrix. We suppose Λ is non-singular, three times differentiable in S, and that $\Lambda^{-1}(t, S)$ and all derivatives of Λ have at most linear growth in S. Writing $\mu(t, S) = (\mu^1(t, S), \dots, \mu^d(t, S))'$ we also suppose μ is three times differentiable in S with all derivatives having at most linear growth in S.

We suppose the stocks pay nc dividends. However, suppose there is a bond S_t^0 with a fixed interest rate r, so $S_t^0 = e^{rt}$. The discounted stock price vector $\xi_t = (\xi_t^1, \dots, \xi^d)'$ is then $\xi_t := e^{-rt} S_t$ so

$$d\xi_t^i = \xi_t^i \Big((\mu^i(t, e^{rt}\xi_t) - r)dt + \sum_{j=1}^{d} \lambda_{ij}(t, e^{rt}\xi_t) dw_t^j \Big). \tag{4.1}$$

Writing

$$\Delta_t = \Delta(t, \xi_t) = \begin{pmatrix} \xi_t^1 & & 0 \\ & \ddots & \\ 0 & & \xi_t^d \end{pmatrix}$$

and $\rho = (r, r, \dots, r)'$ equation (4.1) can be written

$$d\xi_t = \Delta_t((\mu - \rho)dt + \Lambda_t dw). \tag{4.2}$$

As in Section 2, there is a flow of diffeomorphi·.ns $x \to \xi_{s,t}(x)$ associated with this system, and their non-singular Jacobians $D_{s,t}$.

In the terminology of Harrison and Pliska, [9], the return process $Y_t = (Y_t^1, \dots, Y_t^d)$ is here given by

$$dY = (\mu - \rho)dt + \Lambda dw. \tag{4.3}$$

The drift term in (4.3) can be removed by applying the Girsanov change of measure. Write $\eta(t, S) = \Lambda(t, S)^{-1}(\mu(t, S) - \rho)$ and define the martingale M by

$$M_t = 1 - \int_0^t M_s \eta(s, S)' dw_s.$$

Then

$$M_t = \exp\Big(- \int_0^t \eta' dw_s - \frac{1}{2} \int_0^t |\eta_s|^2 ds \Big)$$

is the Radon–Nikodym derivative of a probability measure $\widetilde{P}$. Furthermore, under $\widetilde{P}$, $\widetilde{w}_t = w_t + \int_0^t \eta(s, S)' ds$ is a standard Brownian motion. Consequently, under $\widetilde{P}$

$$dY_t = \Lambda_t d\widetilde{w}_t$$

and

$$d\xi_t = \Delta_t \Lambda_t d\widetilde{w}_t. \tag{4.4}$$

Therefore, the discounted stock price process ξ is a martingale under $\widetilde{P}$ so $\widetilde{P}$ is a 'risk-neutral' measure.

Consider a function $\overline{\psi} : R^d \to R$, where $\overline{\psi}$ is twice differentiable and $\overline{\psi}$ and $\overline{\psi}_x$ are of at most linear growth in x. For some future time $T > t$ we shall be interested in finding the current price (i.e., current valuation at time t), of a contingent claim of the form $\overline{\psi}(S_T)$. It is convenient to work with the discounted stock price, so we consider equivalently the current value of

$$\psi(\xi_T) := \overline{\psi}(e^{rT}\xi_T).$$

ψ has linear growth, so we may define the square integrable $\widetilde{P}$ martingale N by

$$N_t = \widetilde{E}[\psi(\xi_T) \mid F_t], \qquad 0 \le t \le T.$$

As in Harrison and Pliska, [9], if we can express N in the form

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \phi(s)' d\xi_s$$

then $\phi = (\phi^1, \ldots, \phi^d)'$ is a hedge portfolio that generates the contingent claim. However, we can apply Theorem 3.1 to derive immediately:

THEOREM 4.1.

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \phi(s)' d\xi_s$$

where

$$\phi(s) = \widetilde{E}\Big[\int_s^T \eta_\xi(u, e^{ru}\xi_{0,u}(x_0))D_{0,u}(x_0)d\widetilde{w}_u \cdot \psi(\xi_{0,T}(x_0)) \\ + \psi_\xi(\xi_{0,T}(x_0))D_{0,T}(x_0) \mid F_s\Big] D_{0,s}^{-1}(x_0).$$

PROOF. From Theorem 3.1, under measure $\widetilde{P}$

$$N_t = \widetilde{E}[\psi(\xi_T)] + \int_0^t \gamma_s d\widetilde{w}_s$$

where

$$\gamma_s = \widetilde{E}\Big[\int_s^T \eta_\xi D_{0,u}(x_0)d\widetilde{w}_u \cdot \psi(\xi_{0,T}(x_0)) \\ + \psi_\xi(\xi_{0,T}(x_0))D_{0,T}(x_0) \mid F_s\Big] D_{0,s}^{-1}(x_0)\Delta(\xi_{0,s}(x_0))\Lambda_s.$$

Because $d\xi_t = \Delta_t \Lambda_t d\widetilde{w}_t$, $\phi(s)$ has the stated form.

REMARKS 4.2. Note that if η is not a function of ξ, (which is certainly the situation in the usual log-normal case where μ and Λ are constant), η_ξ is zero and the first term in ϕ vanishes.

The bond component ϕ^0 in the portfolio is given by

$$\phi_t^0 = N_t - \sum_{i=1}^d \phi_t^i \xi_t^i, \qquad 0 \le t \le T$$

and N_t is the price associated with the contingent claim at time t.

5. Examples.

Stock price dynamics for which the hedging policy ϕ can be evaluated in closed form appear hard to find. However, if we consider a vector of log-normal stock prices we can re-derive the Black-Scholes results. Suppose, therefore, that the vector of stock prices $S = (S^1, \dots, S^d)'$ evolves according to the equations

$$dS_t^i = S_t^i\Big(\mu^i dt + \sum_{j=1}^{d} \lambda_{ij} dw_t^j\Big) \tag{5.1}$$

where $\mu = (\mu^1, \dots, \mu^d)$ and $\Lambda = (\lambda_{ij})$, are constant. The discounted stock price ξ is then given by (4.2).

Consider a contingent claim which consists of d European call options with expiry dates $T_1 \le T_2 \le \cdots \le T_d$ and exercise prices $c_1, \dots, c_d$, respectively. Then

$$\psi(T_1, \dots, T_d) = \sum_{k=1}^{d} \psi^k(\xi_{0,T_k}(x_0)) = \sum_{k=1}^{d} (\xi_{0,T_k}^k(x_0) - c_k e^{-rT_k})^+.$$

The ψ^k are convex functions, so applying the generalized Itô differentiation rule of [10] Theorem 3.1 is valid as noted in Remark 3.2 with

$$\psi_\xi^k = (0, \dots, 0, I\{\xi_{0,T_k}^k > c_k e^{-rT_k}\}, 0, \dots, 0).$$

From (5.1) we see that the Jacobian $D_{0,t}$ is just the diagonal matrix

$$D_{0,t} = \begin{bmatrix} \exp\Big\{\sum_{j=1}^{d} \lambda_{1j}\widetilde{w}_t^j - \frac{1}{2}\, a_{11}t\Big\} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \exp\Big\{\sum_{j=1}^{d} \lambda_{dj}\widetilde{w}_t^j - \frac{1}{2}\, a_{dd}t\Big\} \end{bmatrix}$$

and its inverse is

$$D_{0,t}^{-1} = \begin{bmatrix} \exp\Big\{-\Big(\sum_{j=1}^{d} \lambda_{1j}\widetilde{w}_t^j - \frac{1}{2}\, a_{11}t\Big)\Big\} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \exp\Big\{-\Big(\sum_{j=1}^{d} \lambda_{dj}\widetilde{w}_t^j - \frac{1}{2}\, a_{dd}t\Big)\Big\} \end{bmatrix}.$$

(The explicit, exponential form of the solution shows $D_{0,t}$ is independent of x_0.) Thus, the trading strategy ϕ_k that generates the contingent claim $\psi^k(\xi_{T_k})$ is

$$\begin{aligned}\phi_k(s)' &= \widetilde{E}[\psi^k_\xi(\xi_{0,T_k}(x_0))D_{0,T_k} \mid \mathcal{F}_s]D_{0,s}^{-1} \\ &= \Big(0,\dots,0,\widetilde{E}\Big[I\{\xi_{0,T_k} > c_k e^{-rT_k}\}\exp\Big\{\sum_{j=1}^d \lambda_{kj}(\widetilde{w}^j_{T_k} - \widetilde{w}^j_s) \\ &\qquad -\frac{1}{2}\, a_{kk}(T_k - s)\Big\} \mid \mathcal{F}_s\Big],0,\dots,0\Big),\end{aligned}$$

for $0 \le s \le T_k$. Note that $\phi_k(s) = 0$ for $s > T_k$, i.e., $\phi_k(s)$ stops at T_k. However, from (5.1),

$$\xi^k_{0,T_k}(x_0) = x^k_0 \exp\Big\{\sum_{j=1}^d \lambda_{kj}\widetilde{w}^j_{T_k} - \frac{1}{2}\, a_{kk}T_k\Big\} > c_k e^{-rT_k}$$

iff

$$\sum_{j=1}^d \lambda_{kj}\widetilde{w}^j_{T_k} > \log\Big(\frac{c_k}{x^k_0}\Big) + \Big(\frac{1}{2}\, a_{kk} - r\Big)T_k = \alpha_k, \quad \text{say;} \tag{5.2}$$

that is, iff

$$\sum_{j=1}^d \lambda_{kj}(\widetilde{w}^j_{T_k} - \widetilde{w}^j_s) > \alpha_k - \sum_{j=1}^d \lambda_{kj}\widetilde{w}^j_s.$$

Now, under $\widetilde{P}$, $\sum_{j=1}^d \lambda_{kj}(\widetilde{w}^j_{T_k} - \widetilde{w}^j_s)$ is normally distributed with mean zero, variance $a_{kk}(T_k - s)$, and is independent of $\mathcal{F}_s$. Therefore, the nonzero component of $\phi_k(s)$ is

$$\begin{aligned}&\int_{\alpha_k - \sum_{j=1}^d \lambda_{kj}\widetilde{w}^j_s}^{\infty} \exp\{x - \frac{1}{2}\, a_{kk}(T_k - s)\}\exp\Big\{\frac{-x^2}{2a_{kk}(T_k - s)}\Big\}\frac{dx}{\sqrt{2\pi a_{kk}(T_k - s)}} \\ &= \int_{\alpha_k - \sum_{j=1}^d \lambda_{kj}\widetilde{w}^j_s}^{\infty} \exp\Big\{\frac{-[x - a_{kk}(T_k - s)]^2}{2a_{kk}(T_k - s)}\Big\}\frac{dx}{\sqrt{2\pi a_{kk}(T_k - s)}} \\ &= \int_{\frac{\alpha_k - \sum \lambda_{kj}\widetilde{w}^j_s - a_{kk}(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}}^{\infty} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\ &= \Phi\Big(\frac{-\alpha_k + \sum \lambda_{kj}\widetilde{w}^j_s + a_{kk}(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\Big).\end{aligned}$$

Again from (1), $\sum_{j=1}^{d} \lambda_{kj}\widetilde{w}_s^j = \log\left(\frac{\xi_{0,s}^k(x_0)}{x_0^k}\right) + \frac{1}{2}\, a_{kk}s$, which together with (5.2) gives

$$\phi_k(s) = \left(0,\ldots,0,\Phi\left(\frac{\log\left(\frac{\xi_{0,s}^k(x_0)}{c_k}\right) - \frac{1}{2}\, a_{kk}(T_k - s) + rT_k}{\sqrt{a_{kk}(T_k - s)}}\right),0,\ldots,0\right)',$$

or, in terms of the (nondiscounted) price S_s^k,

$$\phi_k(s) = \left(0,\ldots,0,\Phi\left(\frac{\log\left(\frac{S_s^k}{c_k}\right) - (\frac{1}{2}\, a_{kk} - r)(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\right),0,\ldots,0\right)', \tag{5.3}$$

$0 \le s \le T_k$. Therefore, the trading strategy ϕ generating $\psi(T_1,\ldots,T_k) = \sum_{k=1}^{d} \psi^k(\xi_{T_k})$ can be written, with a slight abuse of notation, as $\phi(s) = (\phi_1(s),\ldots,\phi_d(s))'$, where

$$\phi_k(s) = I_{\{s \le T_k\}}\Phi\left(\frac{\log\left(\frac{S_t^k}{c_k}\right) - (\frac{1}{2}\, a_{kk} - r)(T_k - s)}{\sqrt{a_{kk}(T_k - s)}}\right). \tag{5.4}$$

Finally, we calculate the price of the claim $\widetilde{E}[\psi(T_1,\ldots,T_d)] = \sum_{k=1}^{d} \widetilde{E}[\psi^k(\xi_{T_k})]$ similarly:

$$\sum_{k=1}^{d} \widetilde{E}[\psi^k(\xi_{T_k})] = \sum_{k=1}^{d} \widetilde{E}(\xi_{T_k}^k - c_k e^{-rT_k})^+$$

$$= \sum_{k=1}^{d} \widetilde{E}\Big[I\Big\{\sum_{j=1}^{d} \lambda_{kj}\widetilde{w}_{T_k}^j > \alpha_k\Big\}\Big(Z_0 \exp\Big\{\sum_{j=1}^{d} \lambda_{kj}\widetilde{w}_{T_k}^j - \frac{1}{2}\, a_{kk}T_k\Big\} - c_k e^{-rT_k}\Big)\Big]$$

$$= \sum_{k=1}^{d} S_0^k \Phi\left(\frac{\log\left(\frac{S_0^k}{c_k}\right) + (\frac{1}{2}\, a_{kk} + r)T_k}{\sqrt{a_{kk}T_k}}\right)$$

$$- c_k e^{-rT_k}\Phi\left(\frac{\log\left(\frac{S_0^k}{c_k}\right) + (\frac{1}{2}\, a_{kk} + r)T_k}{\sqrt{a_{kk}T_k}} - \sqrt{a_{kk}T_k}\right)$$

(where we have used $\xi_0^k = S_0^k,\ k = 1,\ldots,d$). When $d = 1$ the above result reduces to the well-known Black–Scholes formula.

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FILTERING FOR A LOGISTIC EQUATION

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1. INTRODUCTION

Filtering is a mathematical theory of estimating a "signal" from noisy observations. It has had striking successes in many areas of engineering. For example, the linear Kalman filter described in Section 2 below was derived in 1960 and is credited with a large role in the U.S. space program. Once the theory and techniques are more widely known it is likely that filtering will have important applications in other areas. Applications of filtering to biological problems can be found in Refs [1–4], for example.

In Section 2 the form of the linear Kalman filter is derived. The analogous equation in the nonlinear situation is obtained in Section 3. This equation has a quadratic term, so in Section 4 the Zakai equation for the unnormalized density is established. Finally, in Section 5, a logistic equation with some noise is discussed. Using a technique of Kunita [5], it is shown how the unnormalized conditional estimate can be expressed without using stochastic integrals with respect to the observation process. Fuller details of the ideas described can be found in Refs [6, 7].

2. THE KALMAN FILTER

The basic idea of the model is that there is a signal process $\{x_t\}$, which cannot be observed directly, and a related observation process $\{y_t\}$. The object is to obtain the "best estimate" of x_t, given the history of y up to time t. To illustrate the ideas we first discuss the linear Kalman filter. For simplicity the processes will be one-dimensional.

Suppose the signal process is given by the following linear stochastic differential equation:

$$\mathrm{d}x_t = Ax_t\,\mathrm{d}t + C\,\mathrm{d}W_t,$$

where W_t is a Brownian motion. Then

$$x_t = x_0 + \int_0^t Ax_s\,\mathrm{d}s + CW_t.$$

The observation process will be assumed to be of the form

$$\mathrm{d}y_t = Hx_t\,\mathrm{d}t + \mathrm{d}B_t,$$

where B_t is a second Brownian motion independent of W. Assume $x_0 \sim N(0, P_0)$ and $y_0 = 0$.

The filtering problem is to calculate a recursive expression for the "best estimate" of x_t, given $\{y_s : s \leqslant t\}$.

Write $Y_t = \sigma\{y_s : s \leqslant t\}$ for the σ-field generated by y_s up to time t and

$$\hat{x}_t = E[x_t | Y_t].$$

Then $\hat{x}_t$ is the "best estimate" in the sense that it minimizes $E(x_t - z)^2$ over all Y_t-measurable, square integrable random variables z.

Calculating $\hat{x}_t$ is a Hilbert space projection problem. Suppose (Ω, F, P) is the probability space on which our random variables are defined. Consider $L_0^2(\Omega)$, the space of square integrable random variables of zero mean. For $X, Y \in L_0^2(\Omega)$, the inner product is just $E(XY)$.

The process v has the following properties:

(i) v_t is a Y_t Brownian motion, i.e. v_t is a Y_t-martingale and $\langle v \rangle_t = t$, where $\langle v \rangle$ is the unique predictable process such that $v^2 - \langle v \rangle$ is a martingale.
(ii) All Y_t-martingales are stochastic integrals with respect to v_t, i.e. if M_t is a Y-martingale, then there is a process g_t which is Y_t-predictable and

$$M_t = \int_0^t g_s \, \mathrm{d}v_s.$$

Suppose the signal process is an F_t-semimartingale ξ, given by

$$\xi_t = \xi_0 + \int_0^t \alpha_s \, \mathrm{d}s + \eta_t.$$

Here α is an F-adapted process such that

$$E \int_0^T \alpha_s^2 \, \mathrm{d}s < \infty,$$

ξ_0 is an F_0-measurable random variable with $E\xi_0^2 < \infty$, and η_t is a square integrable F_t-martingale. There is a unique predictable process $\langle \eta, B \rangle$ such that $\eta_t B_t - \langle \eta, B \rangle_t$ is a martingale. We shall suppose this is of the form

$$\langle \eta, B \rangle_t = \int_0^t \beta_s \, \mathrm{d}s.$$

Theorem 1. *Write $\hat{\xi}_t = E[\xi_t / Y_t]$ for the filtered estimate of ξ_t, given Y_t. Then*

$$\hat{\xi}_t = \hat{\xi}_0 + \int_0^t \hat{\alpha}_s \, \mathrm{d}s + \int_0^t (\widehat{\xi_s h}(x_s) - \hat{\xi}_s \hat{h}(x_s) + \hat{\beta}_s) \, \mathrm{d}v_s.$$

Proof. Write

$$\mu_t = \hat{\xi}_t - \hat{\xi}_0 - \int_0^t \hat{\alpha}_s \, \mathrm{d}s.$$

Then it is easy to check that μ is a Y_t-martingale. Consequently, by property (ii) above there is a process g_s, such that

$$\mu_t = \int_0^t g_s \, \mathrm{d}v_s.$$

Again we wish to determine g. To do this recall

$$\xi_t = \xi_0 + \int_0^t \alpha_s \, \mathrm{d}s + \eta_t, \tag{1}$$

$$\hat{\xi}_t = \hat{\xi}_0 + \int_0^t \hat{\alpha}_s \, \mathrm{d}s + \int_0^t g_s \, \mathrm{d}v_s, \tag{2}$$

$$y_t = \int_0^t h(\xi_s) \, \mathrm{d}s + B_t \tag{3}$$

and

$$y_t = \int_0^{\cdot} \hat{h}(\xi_s) \, \mathrm{d}s + v_t. \tag{4}$$

From equations (1) and (3), using the Ito rule:

$$\xi_t y_t = \int_0^t \xi_s [h(\xi_s) \, \mathrm{d}s + \mathrm{d}B_s] + \int_0^t y_s (\alpha_s \, \mathrm{d}s + \mathrm{d}\eta_s) + \int_0^t \beta_s \, \mathrm{d}s.$$

Conditioning each side on Y_t, we see that

$$E[\xi_t y_t | Y_t] = y_t\hat{\xi}_t = \int_0^t [\widehat{\xi_s h}(\xi_s) + y_s\hat{\alpha}_s + \hat{\beta}_s]\,\mathrm{d}s + N_t^1, \tag{5}$$

where N_t^1 is a Y_t-martingale. However, from equations (2) and (4), using the Ito rule:

$$\begin{aligned} y_t\hat{\xi}_t &= \int_0^t \hat{\xi}_s\hat{h}(\xi_s)\,\mathrm{d}s + \int_0^t \hat{\xi}_s\,\mathrm{d}v_s + \int_0^t y_s(\hat{\alpha}_s\,\mathrm{d}s + g_s\,\mathrm{d}v_s) + \int_0^t g_s\,\mathrm{d}s \\ &= \int_0^t [\hat{\xi}_s\hat{h}(\xi_s) + y_s\hat{\alpha}_s + g_s]\,\mathrm{d}s + N_t^2, \end{aligned} \tag{6}$$

where N_t^2 is a Y_t-martingale. The decompositions of $y\hat{\xi}_t$ in equations (5) and (6) must be the same, so

$$N_t^1 = N_t^2$$

and

$$\widehat{\xi_s h}(\xi_s) + y_s\hat{\alpha}_s + \hat{\beta}_s = \hat{\xi}_s\hat{h}(\xi_s) + y_s\hat{\alpha}_s + g_s.$$

This gives $g_s = \widehat{\xi_s h}(\xi_s) - \hat{\xi}_s\hat{h}(\xi_s) + \hat{\beta}_s$, and the result follows. End of proof.

Suppose that x_t is the solution of the stochastic differential equation

$$\mathrm{d}x_t = f(t, x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}w_t.$$

That is,

$$x_t = x_0 + \int_0^t f(s, x_s)\,\mathrm{d}s + \int_0^t \sigma(x_s)\,\mathrm{d}w_s.$$

If F is a C^2 function, the Ito differential rule tells us that

$$\begin{aligned} F(x_t) = F(x_0) &+ \int_0^t F_x(x_s) f(s, x_s)\,\mathrm{d}s \\ &+ \int_0^t F_x(x_s)\sigma(x_s)\,\mathrm{d}w_s + \frac{1}{2}\int_0^t F_{xx}(x_s)\sigma(x_s)^2\,\mathrm{d}s. \end{aligned} \tag{7}$$

Let us use the notation

$$\Pi_t(F) = E[f(x_t)|Y_t].$$

Then Π_t can be thought of as the conditional distribution of x_t, given Y_t, so that

$$\Pi_t(F) = \int F(x)\Pi_t(\mathrm{d}x).$$

Suppose w is independent of the observation noise B. Applying Theorem 1 to the semimartingale $F(x_t)$ gives

$$\Pi_t(F) = \Pi_0(F) + \int_0^t \Pi_s(\Gamma F)\,\mathrm{d}s + \int_0^t [\Pi_s(Fh) - \Pi_s(F)\Pi_s(h)]\,\mathrm{d}v_s. \tag{8}$$

Here,

$$\Gamma F(x) = f(s, x)F_x(x) + \tfrac{1}{2}\sigma(x)^2 F_{xx}(x).$$

This equation gives an infinite dimensional recursive equation for the filtered estimate $\Pi_t(F)$.

If we consider the case $f(s, x) = A$, $\sigma(x_s) = C$, $F(x) = x$ and $h(x) = Hz$, equation (8) reduces to the Kalman filter derived in Section 2.

4. THE UNNORMALIZED FILTERING EQUATION

There are two difficulties with equation (8). Firstly, it is in some sense quadratic in Π and, secondly, it is driven by the innovations process ν. Suppose we are considering processes defined on $[0, T]$. Define a new probability measure P_0 on (Ω, F) by

$$\frac{dP_0}{dP} = \exp\left[-\int_0^T h(x_s)\, dB_s - \frac{1}{2}\int_0^T h^2(x_s)\, ds\right].$$

If h is of linear growth, say $|h(x)| \leqslant K(1 + |x|)$, then P_0 is a probability measure and under P_0, y_t is actually a Brownian motion. This is a result of Girsanov's theorem [6, Theorem 13.14]. Write E_0 for expectation with respect to P_0 and

$$\Lambda_t = \exp\left[\int_0^t h(x_s)\, dy_s - \frac{1}{2}\int_0^t h^2(x_s)\, ds\right].$$

Then, using a Baye's type theorem,

$$\begin{aligned}\Pi_t(F) &= E[F(x_t)|Y_t] \\ &= \frac{E_0[\Lambda_t F(x_t)|Y_t]}{E_0[\Lambda_t|Y_t]} \\ &= \frac{\sigma_t(F)}{\sigma_t(1)} \quad \text{say,}\end{aligned}$$

where $\sigma_t(F) = E_0[\Lambda_t F(x_t)|Y_t]$ is an unnormalized conditional distribution of $F(x_t)$.

We first obtain a semimartingale expression for $\sigma_t(1)$. Using the Ito differentiation rule:

$$d\Lambda_t = h(x_t)\Lambda_t\, dy_t.$$

That is,

$$\Lambda_t = 1 + \int_0^t h(x_s)\Lambda_s\, dy_s \tag{9}$$

so Λ_t is a (F_t, P_0)-martingale. Consequently, $\hat{\Lambda}_t = E_0[\Lambda_t|Y_t]$ is a Y_t-martingale so there is some Y_t-predictable process γ_t, such that

$$\hat{\Lambda}_t = 1 + \int_0^t \gamma_s\, dy_s. \tag{10}$$

To determine γ_s consider, using equation (9),

$$y_t\Lambda_t = \int_0^t \Lambda_s\, dy_s + \int_0^t y_s h(x_s)\Lambda_s\, dy_s + \int_0^t \Lambda_s h(x_s)\, ds.$$

Conditioning on Y_t under measure P_0, we have

$$E_0[y_t\Lambda_t|Y_t] = y_t\hat{\Lambda}_t = \int_0^t \widehat{\Lambda_s h}(x_s)\, ds + M_t^1, \tag{11}$$

where M_t^1 is a (Y_t, P_0)-martingale. However, using equation (10),

$$\begin{aligned}y_t\hat{\Lambda}_t &= \int_0^t y_s\gamma_s\, dy_s + \int_0^t \hat{\Lambda}_s\, dy_s + \int_0^t \gamma_s\, ds \\ &= \int_0^t \gamma_s\, ds + M_t^2.\end{aligned} \tag{12}$$

Again, the decompositions (11) and (12) must be the same, so

$$M_t^1 = M_t^2$$

and

$$\gamma_s = \widehat{\Lambda_s h}(x_s) = E_0[\Lambda_s h(x_s) | Y_s].$$

Using Baye's rule again, we see that

$$\gamma_s = \hat{\Lambda}_s \Pi_s(h(x_s)).$$

Therefore,

$$\hat{\Lambda}_t = 1 + \int_0^t \hat{\Lambda}_s \Pi_s(h(x_s))\, \mathrm{d}y_s. \tag{13}$$

(Note $\widehat{\ }$ denotes conditioning under measure P_0, while Π denotes conditioning under the original measure P.) Now equation (13) has the unique solution

$$\begin{aligned}\hat{\Lambda}_t &= \exp\left[\int_0^t \Pi_s(h)\, \mathrm{d}y_s - \frac{1}{2}\int_0^t \Pi_s(h)^2\, \mathrm{d}s\right]\\ &= \sigma_t(1).\end{aligned}$$

Recall $\sigma_t(F) = \sigma_t(1)\Pi_t(F)$. Forming the product of equations (8) and (13) therefore, and using the Ito rule, we have the following result:

Theorem 2. *$\sigma_t(F)$ satisfies the "Zakai equation",*

$$\sigma_t(F) = \sigma_0(F) + \int_0^t \sigma_t(\Gamma F)\, \mathrm{d}s + \int_0^t \sigma_t(hF)\, \mathrm{d}y_s, \tag{14}$$

with initial condition

$$\sigma_0(F) = \Pi_0(F) = E[F(x_0)].$$

This equation is linear in σ and is driven by the observation process y. There is a one-to-one correspondence between solutions of the Zakai equation and solutions of the nonlinear filtering equation (8): whenever $\sigma_t(F)$ satisfies equation (14), then $\sigma_t(F)/\sigma_t(1)$ satisfies equation (8), and whenever $\Pi_t(F)$ satisfies equation (8),

$$\Pi_t(F)\exp\left[\int_0^t \Pi_s(h)\, \mathrm{d}y_s - \frac{1}{2}\int_0^t \Pi_s(h)^2\, \mathrm{d}s\right]$$

satisfies equation (13).

5. EXAMPLE

Suppose the state of the system under investigation is described by the following logistic-type equation:

$$\mathrm{d}N_t = \lambda N_t(1 - \alpha N_t)\, \mathrm{d}t + \epsilon\, \mathrm{d}w_t.$$

That is, N satisfies the usual logistic-type equation with a small amount of noise represented by $\epsilon\, \mathrm{d}w$.

The observation process will be of the form

$$\mathrm{d}y_t = kN_t\, \mathrm{d}t + \mathrm{d}B_t.$$

Again, if noise were not present in the observation process we would have

$$N_t = k^{-1}\frac{\mathrm{d}y_t}{\mathrm{d}t},$$

so that N_t could be determined from the rate of growth of the observation process. Write $\Pi_t(N) = E[N_t | Y_t]$, so the innovations process is

$$v_t = y_t - \int_0^t k\Pi_s(N)\,\mathrm{d}s.$$

Suppose the observation noise B and state, or signal, noise w are independent. Then equation (8) for the filtered estimate $\Pi_t(N)$ gives

$$\Pi_t(N_t) = \Pi_0(N_0) + \int_0^t \Pi_s(\lambda N_s(1-\alpha N_s))\,\mathrm{d}s + k\int_0^t (\Pi_s(N_s^2) - \Pi_s(N_s)^2)\,\mathrm{d}v_s.$$

Equation (13) for the unnormalized filtered estimate $\sigma_t(N)$ gives

$$\sigma_t(N_t) = \Pi_0(N_0) + \int_0^t \sigma_s(\lambda N_s(1-\alpha N_s))\,\mathrm{d}s + k\int_0^t \sigma_s(N_s^2)\,\mathrm{d}y_s.$$

Again consider a C^2 function F so that, using the Ito rule,

$$F(N_t) = F(N_0) + \int_0^t AF(N_s)\,\mathrm{d}s + \epsilon\int_0^t F_N(N_s)\,\mathrm{d}w_s.$$

Here,

$$AF(N_s) = \lambda N_s(1-\alpha N_s)F_N(N_s) + \frac{\epsilon^2}{2}F_{NN}(N_s).$$

Therefore, equation (8) gives

$$\Pi_t(F) = \Pi_0(F) + \int_0^t \Pi_s(AF)\,\mathrm{d}s + k\int_0^t [\Pi_s(N_sF(N_s)) - \Pi_s(N_s)\Pi_s(F(N_s))]\,\mathrm{d}v_s.$$

If Π_t is given by a conditional density $\hat{p}$, then

$$\Pi_t(F) = \int_R F(x)\hat{p}(t,x,y)\,\mathrm{d}x.$$

Consequently, $\hat{p}$ is given by the equation

$$\hat{p}(t,x,y) = \hat{p}(0,x,y) + \int_0^t A^*\hat{p}(s,x,y)\,\mathrm{d}s + k\int_0^t \hat{p}(s,x,y)(x-\hat{x}_s)\,\mathrm{d}v_s.$$

Here,

$$A^*\phi(x) = -\frac{\partial}{\partial x}[\lambda x(1-\alpha x)\phi(x)] + \frac{\epsilon^2}{2}\frac{\partial^2\phi(x)}{\partial x^2}$$

is the adjoint of A. This equation is not linear in $\hat{p}$ because

$$\hat{x}_s = \int_R x\hat{p}(s,x,y)\,\mathrm{d}s.$$

However, if we consider the related unnormalized conditional density q, given by

$$q(t,x) = \hat{\Lambda}_t\hat{p}(t,x),$$

then

$$\sigma_t(F) = \int_R F(x)q(t,x)\,\mathrm{d}s.$$

Furthermore, q is given by the linear equation

$$q(t,x)=\hat{p}(0,x,y)+\int_0^t A^*q(s,x)\,\mathrm{d}s+k\int_0^t xq(s,x)\,\mathrm{d}y_s,$$

which has y as input.

Finally, let us consider again the Zakai equation for $\sigma_t(F)$:

$$\sigma_t(F)=\sigma_0(F)+\int_0^t \sigma_s(AF)\,\mathrm{d}s+k\int_0^t \sigma_s(N_sF(N_s))\,\mathrm{d}y_s.$$

In terms of Stratonovich integrals this is

$$\sigma_t(F)=\sigma_0(F)+\int_0^t \sigma_s\left(AF(N_s)-\frac{k}{2}N_s^2F(N_s)\right)\mathrm{d}s+k\int_0^t \sigma_s(N_sF(N_s))\circ\mathrm{d}y_s.$$

Consider the following operators defined on functions $F(N)$:

$$\begin{aligned}L(t)F(N)&=AF(N)-\frac{k}{2}N^2F(N)\\&=\lambda N(1-\alpha N)\frac{\partial F}{\partial N}+\frac{c^2}{2}\frac{\partial^2F}{\partial N^2}-\frac{k}{2}N^2F(N),\end{aligned}$$

$$\mu_tF(N)=F(N)\exp(Ny_t)$$

and

$$\mu_t^{-1}F(N)=F(N)\exp(-Ny_t).$$

Writing $\Delta(N)$ for $\lambda N(1-\alpha N)$, see that

$$[\mu_tL(t)\mu_t^{-1}]F(N)=\frac{c^2}{2}\frac{\partial^2F}{\partial N^2}+[\Delta(N_t)-c^2y_t]\frac{\partial F}{\partial N}+\left[\frac{c^2y_t^2}{2}-\Delta(N_t)y_t-\frac{k}{2}N^2\right]F(N),$$

so $\mu_tL(t)\mu_t^{-1}$ is a second-order operator.

If we consider $\tilde{N}_t$ as the solution of the system

$$\tilde{N}_t=N_0+\int_0^t[\Delta(\tilde{N}_s)-c^2y_s]\,\mathrm{d}s+cw_t$$

and consider the expectation, given Y_t, of

$$F(\tilde{N}_t)\exp\left\{\int_0^t\left[\frac{c^2y^2}{2}-\Delta(\tilde{N}_s)y_s-\frac{k}{2}\tilde{N}_s^2\right]\mathrm{d}s\right\},$$

we have, writing

$$v_t(F)=E\left(F(\tilde{N}_t)\exp\left\{\int_0^t\left[\frac{c^2y^2}{2}-\Delta(\tilde{N}_s)y_s-\frac{k}{2}\tilde{B}_s^2\right]\mathrm{d}s\right\}\middle|Y_t\right),$$

that

$$v_t(F)=v_0(F)+\int_0^t v_s(\mu_sL(s)\mu_s^{-1})F(N_s)\,\mathrm{d}s.$$

If we now calculate $v_t(\mu_tF)$, we have

$$v_t(\mu,F)=v_0(F)+\int_0^t v_s(\mu_sL(s)\mu_s^{-1})(\mu_sF)\,\mathrm{d}s+k\int_0^t v_s(\mu_s(N_sF(N_s)))\circ\mathrm{d}y_s.$$

That is, the solution of the Zakai equation is given by

$$v_t(\mu_tF)=E\left(\exp(\tilde{N}_ty_t)F(\tilde{N}_t)\exp\left\{\int_0^t\left[\frac{c^2y_s^2}{2}-\Delta(\tilde{N}_s)y_s-\frac{k}{2}\tilde{N}_s^2\right]\mathrm{d}s\right\}\middle|Y_t\right).$$

The advantage of this expression is that it involves no stochastic integrals. The observation trajectory y appears just as a parameter. Also, the operator $\mu_t L(t)\mu_t^{-1}$ differs from $L(t)$ only by terms of less than second order. For details of the method in this section see Ref. [5].

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INTEGRATION BY PARTS FOR POISSON PROCESSES

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Abstract

Using a perturbation of the rate of a Poisson process, and an inverse time change, an integration by parts formula is obtained. This enables a new form of the integrand in a martingale representation result to be obtained.

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Integration by Parts for Poisson Processes

1. Introduction.

In his paper [3] on the Malliavin Calculus Bismut obtains an integration by parts formula for a diffusion by considering a small perturbation of the trajectories and then compensating for this by using a Girsanov change of measure. That is, suppose ξ denotes the original trajectory and ξ^ϵ the perturbation. Let E (resp. E^ϵ) denote expectation with respect to the original measure (resp. the measure after the Girsanov transformation). Then for any bounded, differentiable function c, it is the case that

$$E[c(\xi)] = E^\epsilon[c(\xi^\epsilon)]. \tag{1.1}$$

The left side of this equation is independent of ε and Bismut obtains his integration by parts formula by differentiating in ε and putting $\varepsilon = 0$. Integration by parts formulae for Markov jump processes have been obtained by Bass and Cranston [1], and Bichteler, Gravereaux and Jacod [2]. Again, the variation of the trajectories considered by these authors consists of perturbing the size of the jumps.

A Poisson process is a counting process, and all jumps are of unit size. Consequently, a perturbation of the trajectories of the kind considered in [1] and [2] does not make sense. Instead we consider below a Girsanov change of measure which alters the rate of the Poisson process by a small amount. This is then compensated by considering a time change of the process under the new measure. An identity analogous to (1.1) is obtained and the integration by parts formula follows by differentiating with respect to a parameter ε and putting $\varepsilon = 0$. The case where the function depends only on finitely many jumps is

discussed first, and the general case, for a functional of the Poisson process over the time interval $[0,1]$, is then deduced.

There is a close relation between integration by parts formulae and martingale representation results. It is well known that any uniformly integrable martingale on the sigma fields generated by a Poisson process can be represented as a stochastic integral with respect to the associated martingale. The integrand can be obtained by considering one jump at a time (though the precise form given in equation (2.6) does not appear to be in the literature). What is interesting is that the integration by parts method gives an alternative expression for this integrand, which does involve a derivative of the functional of the process. The equality of these two expressions is verified in the appendix when the functional depends on finitely many jump times. This expression for the integrand is similar to that obtained by Clark [5] for functionals of Brownian motion.

2. Martingale Representation and Time Change.

Let N be a Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with jump times $T_1, \ldots, T_n, \ldots$. We shall write $T_0 = 0$. Let $G(T_1, \ldots, T_n, \ldots)$ be an integrable function of $T_1, \ldots, T_n, \ldots$. Consider the martingale M defined by:

$$M_t := E[G(T_1, \ldots, T_n, \ldots) \mid \mathcal{F}_t]. \tag{2.1}$$

For $n \geq 1$, write

$$\begin{aligned} \ell^{n-1}(T_n) &= M_{T_n} - M_{T_{n-1}} \\ &= E[G \mid \mathcal{F}_{T_n}] - E[G \mid \mathcal{F}_{T_{n-1}}]. \end{aligned} \tag{2.2}$$

From Theorem T9, Chapter 3 of [4], the martingale M defined by (2.1) has the representation:

$$M_t = E[G] + \int_0^t g_s dQ_s \tag{2.3}$$

where $Q_t = N_t - t$,

$$\begin{aligned} g_s &= g^i(s) \\ &= f^i(s - T_i) - e^{s-T_i} \int_{s-T_i}^{\infty} f^i(u) e^{-u} du \end{aligned} \tag{2.4}$$

on $\{T_i \leq s < T_{i+1}\}$, and

$$f^i(s) = E[M_{T_{i+1}} \mid T_1, \ldots, T_i,\ T_{i+1} - T_i = s].$$

Since $\mathcal{F}_{T_i} = \sigma\{T_1, \ldots, T_i\}$, $T_{i+1} - T_i$ is independent of $T_1, \ldots, T_i$, and exponentially distributed, so

$$\begin{aligned} E[I_{\{T_{i+1} > s\}} M_{T_{i+1}} \mid \mathcal{F}_{T_i}] &= \int_0^\infty I_{\{u + T_i \geq s\}} E[M_{T_{i+1}} \mid T_1, \ldots, T_i,\ T_{i+1} - T_i = u] e^{-u} du \\ &= \int_{(s-T_i) \vee 0}^{\infty} f^i(u) e^{-u} du. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we have

$$g^i(s) = E[G \mid T_1, \ldots, T_i,\ T_{i+1} = s] - e^{s-T_i} E[I_{T_{i+1} > s} G \mid \mathcal{F}_{T_i}]. \tag{2.6}$$

From (2.5)

$$\begin{aligned} M_{T_i} &= E[M_{T_{i+1}} \mid \mathcal{F}_{T_i}] \\ &= \int_0^\infty f^i(u) e^{-u} du. \end{aligned} \tag{2.7}$$

Also, ℓ^i can be written as

$$\ell^i(s) = f^i(s - T_i) - M_{T_i}. \tag{2.8}$$

Using (2.4), (2.7) and (2.8), we have

$$g^i(s) = \ell^i(s) + \int_{]T_i,s]} \ell^i(u) e^{s-u} du. \tag{2.9}$$

Throughout the rest of this paper we let $\{u_t,\ t \geq 0\}$ be a real predictable process satisfying:

(i) $\{u_t,\ t \geq 0\}$ is positive and a.s. bounded, $|u_t| \leq B$ a.s. say.

(ii) There exists a bounded interval, say, $[0, b]$, such that $u_s(w) = 0$ if $s \notin [0, b]$, a.s.

For $\varepsilon > 0$, consider the martingale:

$$\begin{aligned} X_t &:= \int_0^t \varepsilon u_s dQ_s \\ &= \sum_{0 \leq s \leq t} \varepsilon u_s \Delta N_s - \int_0^t \varepsilon u_s ds. \end{aligned} \tag{2.10}$$

Define the family of exponentials

$$\begin{aligned} \Lambda_t^\varepsilon &:= \exp(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t) \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \\ &= \prod_{0 \leq s \leq t} (1 + \varepsilon u_s \Delta N_s) \exp\Big(- \int_0^t \varepsilon u_s ds \Big). \end{aligned} \tag{2.11}$$

Then $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ satisfies the equation:

$$\begin{aligned} \Lambda_t^\varepsilon &= 1 + \int_0^t \Lambda_{s-}^\varepsilon dX_s \\ &= 1 + \int_0^t \Lambda_{s-}^\varepsilon \varepsilon u_s dQ_s \end{aligned} \tag{2.12}$$

and $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is a martingale. (See [6].)

LEMMA 2.1. *$\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is a uniformly integrable martingale. Hence $\Lambda_\infty^\varepsilon$ exists and a new probability measure P^ε can be defined by*

$$\frac{dP^\varepsilon}{dP} = \Lambda_\infty^\varepsilon.$$

Proof. It suffices to show that the martingale $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is square integrable.

Recall u vanishes outside the interval $[0, b]$ and $|u_s| < B$ a.s. By (2.12) and Itô's rule,

$$\begin{aligned}(\Lambda_t^\varepsilon)^2 &= 1 + 2\int_0^t \Lambda_{s-}^\varepsilon d\Lambda_s^\varepsilon + \sum_{0 \leq s \leq t} (\Lambda_{s-}^\varepsilon \varepsilon u_s \Delta N_s)^2 \\ &= 1 + 2\int_0^t \Lambda_{s-}^\varepsilon d\Lambda_s^\varepsilon + \int_0^t (\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2 dQ_s + \int_0^t (\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2 ds.\end{aligned}$$

For $0 \leq t \leq b$,

$$\begin{aligned}E[(\Lambda_t^\varepsilon)^2] &= 1 + \int_0^t E[(\Lambda_{s-}^\varepsilon)^2 \varepsilon^2 u_s^2] ds \\ &\leq 1 + \varepsilon^2 B^2 \int_0^t E[(\Lambda_s^\varepsilon)^2] ds.\end{aligned}$$

So by Gronwall's inequality,

$$\begin{aligned}E[(\Lambda_t^\varepsilon)^2] &\leq \exp(\varepsilon^2 B^2 t) \\ &\leq \exp(\varepsilon^2 B^2 b) \qquad 0 \leq t \leq b.\end{aligned}$$

Λ_t^ε is contant for $t > b$. Hence the martingale $\{\Lambda_t^\varepsilon,\ t \geq 0\}$ is square integrable.

$\Lambda_\infty^\varepsilon > 0$ a.s. and $E[\Lambda_\infty^\varepsilon] = 1$ so we can define a new probability measure P^ε by putting

$$\frac{dP^\varepsilon}{dP} = \Lambda_\infty^\varepsilon.$$

Then the process $\{Q_t^\varepsilon\}$ defined by

$$Q_t^\varepsilon := N_t - \int_0^t (1+\varepsilon u_s)ds$$

is an $(\mathcal{F}_t)$ martingale under P^ε (see [7]).

Now define

$$\phi_\varepsilon(t) := \int_0^t (1+\varepsilon u_s)ds. \tag{2.13}$$

Let $\psi_\varepsilon(t) = \phi_\varepsilon^{-1}(t)$. Then $\psi_\varepsilon(\phi_\varepsilon(t)) = t$ so

$$\psi_\varepsilon(t) = \int_0^t \frac{1}{1+\varepsilon u_{\psi_\varepsilon(s)}}\, ds. \tag{2.14}$$

If we let $\mathcal{F}_t^\varepsilon = \mathcal{F}_{\psi_\varepsilon(t)}$, then from Theorem T16, Chapter II of [4], the process $\{N_t^\varepsilon,\ t \geq 0\}$ defined by:

$$N_t^\varepsilon := N_{\psi_\varepsilon(t)} \tag{2.15}$$

is a Poisson process on $(\Omega, \mathcal{F}, (\mathcal{F}_t^\varepsilon), P^\varepsilon)$.

3. Integration by Parts.

Suppose G is a function of the first n jump times $T_1, \ldots, T_n$ of a Poisson process N. Since $\phi_\varepsilon(t) = \psi_\varepsilon^{-1}(t)$, if T_i is the i-th jump time of $\{N_t\}$, then $\phi_\varepsilon(T_i)$ is the i-th jump time of the process $\{N_{\psi_\varepsilon(t)}\}$. Changing the rate of the point process by a Girsanov transformation, and then changing the time scale of the process, we have the following result:

THEOREM 3.1. *Let $G(T_1, \ldots, T_n)$ be bounded with bounded first partial derivatives. Then*

$$E\Big[\Big(\int_0^\infty u_s dQ_s\Big) G(T_1, \ldots, T_n)\Big] = -E\Big[\sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big]. \tag{3.1}$$

Proof. By the results in Section 2, because $N_{\psi_\varepsilon(t)}$ is a Poisson process under P^ε with jump times $\phi_\varepsilon(T_i)$; consequently

$$E[G(T_1,\dots,T_n)] = E^\varepsilon[G(\phi_\varepsilon(T_1),\dots,\phi_\varepsilon(T_n))]$$

$$= E[\Lambda^\varepsilon_\infty G(\phi_\varepsilon(T_1),\dots,\phi_\varepsilon(T_n))] \tag{3.2}$$

where $E^\varepsilon[\ ]$ denotes that expectation is taken with respect to P^ε. Differentiating (3.2) with respect to ε, and then setting $\varepsilon = 0$, we get:

$$E\Big[\Lambda^\varepsilon_{\infty|\varepsilon=0} \frac{d}{d\varepsilon} G(\phi_\varepsilon(T_1),\dots,\phi_\varepsilon(T_n))_{|\varepsilon=0}\Big]$$

$$+ E\Big[\Big(\frac{d}{d\varepsilon} \Lambda^\varepsilon_\infty\Big)_{|\varepsilon=0} G(\phi_\varepsilon(T_1),\dots,\phi_\varepsilon(T_n))_{|\varepsilon=0}\Big] = 0. \tag{3.3}$$

From (2.12) and the definition of $\Lambda^\varepsilon_\infty$,

$$\frac{d\Lambda^\varepsilon_\infty}{d\varepsilon}_{|\varepsilon=0} = \int_0^\infty u_s dQ_s.$$

Noting the definition of ϕ_ε, (3.3) becomes (3.1) and the proof is complete. □

Remark 3.2. Consider a function H of the form $H(T_1 \wedge 1,\dots,T_n \wedge 1)$ where H is bounded and has bounded first derivatives. Applying Theorem 3.1 to $G(T_1,\dots,T_n) = H(T_1 \wedge 1,\dots,T_n \wedge 1)$ and noting that

$$\frac{\partial}{\partial t_i} G(T_1,\dots,T_n) = \frac{\partial}{\partial t_i} H(T_1 \wedge 1,\dots,T_n \wedge 1) I_{T_i \le 1},$$

we have the following:

Corollary 3.3. *If $H(T_1 \wedge 1,\dots,T_n \wedge 1)$ is bounded and has bounded first derivatives, then*

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big) H(T_1 \wedge 1,\dots,T_n \wedge 1)\Big]$$

$$= -E\Big[\sum_{i=1}^n \frac{\partial}{\partial t_i} H(T_1 \wedge 1,\dots,T_n \wedge 1) \int_0^{T_i} u_s ds I_{T_i \le 1}\Big]. \tag{3.4}$$

Remark 3.4. Recall the martingale representation (2.3) and (2.4), or

$$G(T_1, \ldots, T_n) = E[G] + \int_0^{T_n} g_s dQ_s, \tag{3.5}$$

where

$$g_s = g^{i-1}(s) \qquad \text{for} \qquad T_{i-1} \leq s < T_i.$$

If we substitute (3.5) into the left hand side of (3.1), we get

$$E\Big[\Big(\int_0^\infty u_s dQ_s\Big)\Big(E[G] + \int_0^{T_n} g_s dQ_s\Big)\Big] = E\Big[\int_0^{T_n} u_s g_s ds\Big] = E\Big[\int_0^\infty u_s g_s ds\Big] \tag{3.6}$$

where $g_s = 0$ for $s \geq T_n$.

Also, if we consider the measure μ defined by

$$\mu(dt) = \sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n) \delta_{T_i}(dt).$$

Then the right hand side of (3.1) can be written

$$\begin{aligned}
&- E\Big[\sum_{i=1}^n \frac{\partial}{\partial t_i} G(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big] \\
&= -E\Big[\int_0^\infty \int_0^t u_s ds \mu(dt)\Big] \\
&= -E\Big[\int_0^\infty \mu[s, \infty) u_s ds\Big] \\
&= -E\Big[\int_0^\infty \sum_{i=1}^n I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n) u_s ds\Big].
\end{aligned} \tag{3.7}$$

Let $C_s = \sum_{i=1}^n I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1, \ldots, T_n)$. Then there exists a predictable projection C^* of C, such that for each s,

$$C_s^* = E[C_s \mid \mathcal{F}_{s-}] \qquad \text{a.s.}$$

Also for any predictable process $\{u_s,\ s \geq 0\}$,

$$E[u_s C_s] = E[u_s E[C_s \mid \mathcal{F}_{s-}]]$$

$$= E[u_s C_s^*]. \tag{3.8}$$

Let $\mathcal{H}$ be the family of subsets of $[0,\infty) \times \Omega$ of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $s < t$. Recall that the predictable σ-field is generated by $\mathcal{H}$. Taking $u = I_{\{0\}\times F_0}$ or $u = I_{(s,t]\times F}$, then u satisfies the hypothesis in Section 2, so (3.6), (3.7) and (3.8) hold for these u. Also because of (3.8), on comparing (3.6) and (3.7), we have

$$E\Big[\int_0^\infty u_s g_s ds\Big] = -E\Big[\int_0^\infty u_s C_s^* ds\Big]$$

holds for all u which are indicators of sets in $\mathcal{H}$. Since $\mathcal{H}$ generates the predictable σ-field and the processes g and C^* are predictable, therefore we have proved the following result:

PROPOSITION 3.5.

$$g_s = -E\Big[\sum_{i=1}^{n} I_{T_i \geq s} \frac{\partial G}{\partial t_i}(T_1,\ldots,T_n) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \tag{3.9}$$

Now suppose $H(T_1 \wedge 1,\ldots,T_n \wedge 1)$ is bounded with bounded first partial derivatives. From Section 2, H has the representation:

$$H(T_1 \wedge 1,\ldots,T_n \wedge 1) = E[H] + \int_0^{T_n \wedge 1} g_s dQ_s \tag{3.10}$$

where $g_s = g^{i-1}(s)$ for $T_{i-1} \wedge 1 \leq s < T_i \wedge 1$.

An argument similar to the above shows that

$$g_s = -E\Big[\sum_{i=1}^{n} I_{s \leq T_i \leq 1} \frac{\partial H}{\partial t_i}(T_1 \wedge 1,\ldots,T_n \wedge 1) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \tag{3.11}$$

The form of g given in Section 2 and that given in (3.9) are at first sight rather different. A direct proof of their equality is sketched in the Appendix. Next we have the following integration by parts formula:

THEOREM 3.6. *Suppose $G = G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots)$ is a bounded function and its first partial derivatives are all bounded by a constant $K > 0$. Then*

$$E\Big[\Big(\int_0^1 u_s dQ_s\Big) G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots)\Big]$$
$$= -\sum_{i=1}^{\infty} E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) \int_0^{T_i} u_s ds I_{T_i \le 1}\Big]. \quad (3.12)$$

Proof. First note that for each $M > 0$, the partial sum

$$\sum_{i=1}^{M} E[I_{T_i \le 1}] = \sum_{i=1}^{M} P(T_1 \ge i) \le 4e^{-1},$$

so that by hypothesis, the right hand side of (3.12) is finite. For each $n \ge 1$, define

$$H^n(T_1, \dots, T_n) := E[G(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) \mid \mathcal{F}_{T_n}].$$

Then

$$H^n(T_1, \dots, T_n) = E[G(T_1 \wedge 1, \dots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \dots,$$
$$(T_n + S_{n+1} + \dots + S_{n+i}) \wedge 1, \dots) \mid \mathcal{F}_{T_n}]$$
$$= E^S[G(T_1 \wedge 1, \dots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \dots,$$
$$(T_n + S_{n+1} + \dots + S_{n+i}) \wedge 1, \dots)], \quad (3.13)$$

where $S_k = T_k - T_{k-1}$ for $k \geq 1$, and the last expectation E^S in (3.13) is taken only over the random variables $S_{n+1}, \ldots, S_{n+i}, \ldots$, and the $T_1, \ldots, T_n$ are given. From (3.1),

$$E\Big[\Big(\int_0^\infty u_s dQ_s\Big) H^n(T_1, \ldots, T_n)\Big] = -\sum_{i=1}^{n-1} E\Big[\frac{\partial H^n}{\partial t_i}(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big]$$

$$- E\Big[\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) \int_0^{T_n} u_s ds\Big]. \tag{3.14}$$

And from (3.13),

$$\begin{aligned}\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) &= E^S\Big[\frac{\partial}{\partial t_n} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots)\Big] \\ &= E^S\Big[\sum_{i=n}^{\infty} \frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots) I_{T_i \leq 1}\Big] \\ &= \sum_{i=n}^{\infty} E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, (T_n + S_{n+1}) \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big].\end{aligned}$$

Hence

$$\begin{aligned}&E\Big[\frac{\partial H^n}{\partial t_n}(T_1, \ldots, T_n) \int_0^{T_n} u_s ds\Big] \\ &= E\Big[\int_0^{T_n} u_s ds \sum_{i=n}^{\infty} E\Big[\frac{\partial G}{\partial t_i}(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big] \\ &\to 0 \qquad \text{as } n \to \infty\end{aligned} \tag{3.15}$$

by the hypotheses on $\{u_s\}$ and G.

Also for $1 \leq i \leq n-1$,

$$\begin{aligned}&E\Big[\frac{\partial H^n}{\partial t_i}(T_1, \ldots, T_n) \int_0^{T_i} u_s ds\Big] \\ &= E\Big[E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) I_{T_i \leq 1} \mid \mathcal{F}_{T_n}\Big] \int_0^{T_i} u_s ds\Big] \\ &= E\Big[\frac{\partial}{\partial t_i} G(T_1 \wedge 1, \ldots, T_n \wedge 1, \ldots) \int_0^{T_i} u_s ds I_{T_i \leq 1}\Big].\end{aligned} \tag{3.16}$$

Letting $n \to \infty$ in (3.14), because of (3.15) and (3.16), we obtain (3.12). □

We conclude with the following theorem:

THEOREM 3.7. *Suppose M is the right continuous martingale*

$$M_t = E[G(T_1 \wedge 1, \dots T_n \wedge 1, \dots) \mid \mathcal{F}_t].$$

Then

$$M_t = E[G] + \int_0^t g_s dQ_s,$$

where

$$g_s = -E\Big[\sum_{i=1}^{\infty} I_{s \le T_i \le 1} \frac{\partial G}{\partial t_i}(T_1 \wedge 1, \dots, T_n \wedge 1, \dots) \mid \mathcal{F}_{s-}\Big] \qquad \text{a.s.} \tag{3.17}$$

Proof. The proof is similar to the one for Proposition 3.5.

4. Appendix.

We now give a direct proof that the integrands g obtained in sections 2 and 3 are equal.

The idea is to first establish the equivalence on $\{t < T_1\}$ and then to use this to establish the equivalence on $\{T_{i-1} \le t < T_i\}$ for $i > 1$ without any more calculation. First we need a preparatory lemma. Given a bounded measurable function $R(x_1, \dots, x_n)$ define

$$F_R^{(i)}(y_i, \dots, y_n;\ x_1, \dots, x_{i-1}) = R(x_1, \dots, x_{i-1},\ y_i + x_{i-1}, \dots, y_n + x_{i-1}).$$

Notice for later use that

$$\frac{\partial F_R^{(i)}}{\partial y_j} = F^{(i)}_{\partial R/\partial y_j} \qquad \text{for } j \ge i. \tag{4.1}$$

Finally, for i fixed, let $\widetilde{T}_j = T_j - T_{i-1}$ for $j \ge i$ and notice that $(\widetilde{T}_i, \dots, \widetilde{T}_n)$ has the same probability distribution as $(T_1, \dots, T_{n-i+1})$.

LEMMA 4.1. *If $(T_1, \ldots, T_n)$ are the successive jump times of a Poisson process, then*

(a)

$$E[R(T_1, \ldots, T_n) \mid \mathcal{F}_{T_{i-1}}] = E[F_R^{(i)}(\widetilde{T}_i, \ldots, \widetilde{T}_n;\ x_1, \ldots, x_{i-1})]\big|_{(x_1,\ldots,x_{i-1})=(T_1,\ldots,T_{i-1})}$$

(b)

$$E[R(T_1, \ldots, T_n) \mid T_1, \ldots, T_{i-1},\ T_i = t]$$

$$= E[F_R^{(i)}(\widetilde{T}_i, \ldots, \widetilde{T}_n;\ x_1, \ldots, x_{i-1}) \mid \widetilde{T}_i = t - x_{i-1}]\,\big|_{(x_1,\ldots,x_{i-1})=(T_1,\ldots,T_{i-1})}$$

(c)

$$E[R(T_1, \ldots, T_n) \mid \mathcal{F}_t] 1_{\{T_{i-1} \le t < T_i\}}$$

$$= 1_{\{T_{i-1} \le t < T_i\}} E[F_R^{(i)}(\widetilde{T}_i, \ldots, \widetilde{T}_n;\ x_1, \ldots, x_{i-1}) \mid \widetilde{T}_i > t - x_{i-1}]\big|_{(x_1,\ldots,x_{i-1})=(T_1,\ldots,T_{i-1})}.$$

The proof of this lemma is elementary and relies mainly on the independence of $(\widetilde{T}_i, \ldots, \widetilde{T}_n)$ from $T_1, \ldots, T_{i-1}$, and the equality $\mathcal{F}_s \cap \{T_{i-1} \le t < T_i\} = \mathcal{F}_{T_i} \cap \{T_{i-1} \le t < T_i\}$; see [4], Chapter 3, Section 1.

Now we are ready for the proof. Note first that $\mathcal{F}_s = \mathcal{F}_{s-}$, since the sigma algebras are complete.

Step 1 We show equivalence on $\{T_1 > t\}$. A direct calculation and an integration by parts shows that

$$1_{\{T_1 > t\}}(E[G(T_1, \ldots, T_n) \mid T_1 = t] - e^t E[1_{\{T_1 \ge t\}} G(T_1, \ldots, T_n))]$$

$$= 1_{\{T_1 > t\}} E\Big[\sum_1^n \frac{\partial G}{\partial t_j}(T_1, \ldots, T_n) \mid T_1 > t\Big].$$

By part (c) of the lemma and by using the facts that $1_{\{T_1>t\}}$ is $\mathcal{F}_t$ measurable and that $1_{\{T_1>t\}}1_{\{T_j\geq t\}} = 1_{\{T_1>t\}}$ almost sure for $j \geq 1$, we find that this last expression equals

$$1_{\{T_1>t\}}E\Big[\sum_{1}^{n} 1_{\{T_j\geq t\}}\frac{\partial G}{\partial t_j}(T_1,\ldots,T_n) \mid \mathcal{F}_t\Big].$$

This completes Step 1.

Step 2. We prove the equivalence on $\{T_{i-1} \leq t < T_i\}$. First use (a) and (b) of the Lemma to show that

$$\begin{aligned}
&1_{\{T_{i-1}\leq t<T_i\}}\Big(E[G(T_1,\ldots,T_n) \mid T_1,\ldots,T_{i-1},\ T_i = t] - e^{t-T_{i-1}}E[1_{\{T_i>t\}}G \mid \mathcal{F}_{T_{i-1}}]\Big) \\
&\quad = 1_{\{T_{i-1}\leq t\}}1_{\{\widetilde{T}_i>t-x_{i-1}\}}\Big(E[F_G^{(i)}(\widetilde{T}_i,\ldots,\widetilde{T}_n,\ x_1,\ldots,x_{i-1}) \mid \widetilde{T}_i = t - x_{i-1}] \\
&\qquad -e^{t-T_{i-1}}E[F_G^{(i)}(\widetilde{T}_i,\ldots,\widetilde{T}_n;\ x_1,\ldots,x_{i-1})]\Big)\Big|_{(x_1,\ldots,x_{i-1})=(T_1,\ldots,T_{i-1})}.
\end{aligned} \tag{4.2}$$

Now apply Step 1 at each $(x_1,\ldots,x_{i-1})$ with the random vector $(T_1,\ldots,T_n)$ replaced by $(\widetilde{T}_i,\ldots,\widetilde{T}_n)$, and $G(T_1,\ldots,T_n)$ replaced by $F_G^{(i)}(\widetilde{T}_i,\ldots,\widetilde{T}_n;\ x_1,\ldots,x_{i-1})$ and $(x_1,\ldots,x_{i-1})$ fixed but arbitrary. Using (4.1), we find in this way that the expression in (4.2) is equal to

$$1_{\{T_{i-1}\leq t<T_i\}}E\Big[\sum_{j=i}^{n} F^{(i)}_{\partial G/\partial t_j}(\widetilde{T}_i,\ldots,\widetilde{T}_n;\ x_1,\ldots,x_{i-1})\Big]\Big|_{(x_1,\ldots,x_{i-1})=(T_1,\ldots,T_{i-1})}.$$

But from part (c) of the Lemma, this equals

$$\begin{aligned}
&1_{\{T_{i-1}\leq t<T_i\}}E\Big[\sum_{j=i}^{n}\frac{\partial G}{\partial t_j}(T_1,\ldots,T_n) \mid \mathcal{F}_t\Big] \\
&= 1_{\{T_{i-1}\leq t<T_i\}}E\Big[\sum_{j=i}^{n} 1_{\{T_{i-1}\leq t<T_i\}}\ \frac{\partial G}{\partial t_j}(T_1,\ldots,T_n) \mid \mathcal{F}_t\Big] \\
&= 1_{\{T_{i-1}\leq t<T_i\}}E\Big[\sum_{j=1}^{n} 1_{\{T_{i-1}\leq t<T_i\}}1_{\{T_j\geq t\}}\ \frac{\partial G}{\partial t_j}(T_1,\ldots,T_n) \mid \mathcal{F}_t\Big] \\
&= 1_{\{T_{i-1}\leq t<T_i\}}E\Big[\sum_{j=1}^{n} 1_{\{T_j\geq t\}}\ \frac{\partial G}{\partial t_j}(T_1,\ldots,T_n) \mid \mathcal{F}_t\Big].
\end{aligned}$$

The last three equalities are all elementary. This completes the proof.

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The Optimal Control of Diffusions*

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Abstract. Using a differentiation result of Blagovescenskii and Freidlin calculations of Bensoussan are simplified and the adjoint process identified in a stochastic control problem in which the control enters both the drift and diffusion coefficients. A martingale representation result of Elliott and Kohlmann is then used to obtain the integrand in a stochastic integral, and explicit forward and backward equations satisfied by the adjoint process are derived.

1. Introduction

The adjoint process in stochastic control problems has been investigated in several papers. For example, see the works of Bismut [4], [5], Davis and Varaiya [7], Haussmann, [13]-[15], Kushner [16], and the previous papers by Baras *et al.* [1], and Elliott.and Kohlman [11], [12]. In most of these papers the control variable enters only the drift term. However, in an interesting paper [2], Bensoussan considers the case where the control is also present in the diffusion coefficient. By obtaining the Gateaux derivative of the cost function the equation satisfied by the adjoint process is derived. However, a martingale representation result is used and this equation involves an unknown integrand. The contributions of this paper are that, by using results of Blagovescenskii and Freidlin [3] on the differentiability of solutions of stochastic differential equations that depend on a parameter, the

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calculations of Bensoussan can be simplified, and, by applying the expressions obtained by Elliott and Kohlmann in [9] and [10] for the integrand in a stochastic integral, explicit forward and backward equations for the adjoint process are obtained when the optimal control is Markov. In particular, the backward equation is a nonstochastic system of parabolic partial differential equations of a novel form, though it is probably related to the equation obtained in the recent paper by Davis and Spathopoulos [6].

2. Stochastic Dynamics

Assume the state of the system is described by the following equation:

$$dx_t = f(t, x_t, u)\, dt + g(t, x_t, u)\, dw_t, \qquad x_t \in R^d, \quad 0 \le t \le T. \tag{2.1}$$

The control variable u will take values in a compact, convex subset U of some Euclidean space R^k. We assume:

A1. $x_0 \in R^d$ is given.

A2. $f\colon [0, T] \times R^d \times U \to R^d$ is continuous, and continuously differentiable with respect to x, u.

A3. $g\colon [0, T] \times R^d \times U \to R^d \otimes R^n$ is a continuous matrix-valued function, which is continuously differentiable with respect to x, u. The columns of g are denoted by $g^{(k)}$ for $k = 1, \dots, n$.

A4. There is a constant K such that

$$(1 + |x|)^{-1}|f(t, x, u)| + |f_x(t, x, u)| + |f_u(t, x, u)| \le K,$$

$$|g(t, x, u)| + |g_x(t, x, u)| + |g_u(t,x, u)| \le K.$$

A5. $w = (w^1, \dots, w^n)$ is an n-dimensional Brownian motion on a probability space (Ω, F, P).

The right continuous, complete filtration generated by w is denoted by $\{F_t\}$, $0 \le t \le T$.

Notation 2.1. Write $L_F^2[0, T] = \{v \equiv v(t, w) \in L^2(\Omega \times [0, T]; dP \times dt; R^k)$: such that, for a.e. t, $v(t, \cdot) \in L^2(\Omega, F_t; P, R^k)\}$.

Definition 2.2. The set of admissible controls is the set

$$\mathrm{U} = \{v \in L_F^2[0, T]\colon v(t) \in U \text{ a.e. a.s.}\}.$$

Then U is a closed, covex subset of $L_F^2[0, T]$.

Remark 2.3. For each $u \in U$ there is, therefore, a unique strong solution of (2.1). We write $x^u_{s,t}(x)$ if $x \in R^d$, $0 \le s \le t \le T$, for the solution trajectory given by

$$x^u_{s,t}(x) = x + \int_s^t f(r, x^u_{s,r}(x), u_r)\, dr + \int_s^t g(r, x^u_{s,r}(x), u_r)\, dw_r. \tag{2.2}$$

Because $u_t(w)$ depends only on t and w, the result of Blagovenscenskii and Freidlin [3] extends to this situation, so the Jacobian $\partial x^u_{s,t}(x)/\partial x = D^u_{s,t}$ exists and is the solution of

$$D^u_{s,t} = I + \int_s^t f_x(r, x^u_{s,r}(x), u_r) D^u_{s,r}\, dr + \sum_{k=1}^n \int_s^t g^{(k)}_x(r, x^u_{s,r}(x), u_r) D^u_{s,r}\, dw^k_r. \tag{2.3}$$

Here I is the $d \times d$ identity matrix. In fact, if the coefficients f and g are C^k the map $x \to x^u_{s,t}(x)$ is C^{k-1}.

Consider the matrix process H defined by

$$\begin{aligned} H^u_{s,t} = I - \int_s^t H^u_{s,r}\left(f_x(r, x^u_{s,r}(x), u_r) - \sum_{k=1}^n g^{(k)}_x(r, x^u_{s,r}(x), u_r)^2 \right) dr \\ - \sum_{k=1}^n \int_s^t H^u_{s,r} g^{(k)}_x(r, x^u_{s,r}(x), u_r)\, dw^k_r. \end{aligned} \tag{2.4}$$

Using the Ito rule we see $d(H^u_{s,t} D^u_{s,t}) = 0$, and $H^u_{s,s} D^u_{s,s} = I$ so

$$H^u_{s,t} = (D^u_{s,t})^{-1}.$$

Write $\|x^u(x_0)\|_t = \sup_{0 \le s \le t} |x^u_{0,s}(x_0)|$. Then, as in Lemma 2.1 of [15], for any p, $1 \le p < \infty$, using Gronwall's and Jensen's inequalities

$$\|x^u(x_0)\|^p_T \le C\left(1 + |x_0|^p + \left|\int_0^T g(r, x^u_{0,r}(x_0), u_r)\, dw^p_r\right|\right)$$

a.s. for some constant C. Therefore, using Burkholder's inequality and hypothesis A4,

$$\|x^u(x_0)\|_T \quad \text{is in } L^p \qquad \text{for} \quad 1 \le p < \infty.$$

Write

$$\|D^u\|_T = \sup_{0 \le s \le T} |D^u_{0,s}|,$$

$$\|H^u\|_T = \sup_{0 \le s \le T} |H^u_{0,s}|.$$

Then, because f_x and g_x are bounded, an application of Gronwall's, Jensen's, and Burkholder's inequalities again implies

$$\|D^u\|_T \quad \text{and} \quad \|H^u\|_T \quad \text{are in } L^p, \qquad 1 \le p < \infty.$$

Cost 2.4. We assume there is a cost associated with the process, made up of a terminal cost and a running cost,

$$c(x^u_{0,T}(x_0)) + \int_0^T h(r, x^u_{0,r}(x_0), u_r)\, dr.$$

We assume:

A6. $|c(x)| + |c_x| + |c_{xx}(x)| \leq K(1 + |x|^q)$ for some $q < \infty$.

A7. $h: [0, T] \times R^d \times U \to R$ is Borel measurable and continuously differentiable in (x, u).

Furthermore

$$|h_x(t, x, u)| \leq C_1(1 + |x|),$$
$$|h_u(t, x, u)| \leq C_2(1 + |x|).$$

The expected cost if a control $u \in \mathbf{U}$ is used is, therefore,

$$J(u) = E\left[c(x^u_{0,T}(x_0)) + \int_0^T h(r, x^u_{0,r}(x_0), u_r)\, dr\right].$$

We assume there is an optimal control $u^* \in \mathbf{U}$, so that

$$J(u^*) \leq J(u)$$

for all other $u \in \mathbf{U}$.

Notation 2.5. We write x^* for x^{u^*}, $D^*_{0,t}$ for $D^{u^*}_{0,t}$, etc.

3. Differentiability

Assume $u^* \in \mathbf{U}$ is an optimal control. Consider any other control $v \in \mathbf{U}$. Then for $\theta \in [0, 1]$

$$u_\theta(t) = u^*(t) + \theta(v(t) - u^*(t)) \in \mathbf{U}.$$

Now

$$J(u_\theta) \geq J(u^*). \tag{3.1}$$

If the Gateaux derivative $J'(u^*)$ of J, as a functional on the Hilbert space $L^2_F[0, T]$, is well defined, differentiating (3.1) in θ implies

$$\langle J'(u^*), v(t) - u^*(t)\rangle \geq 0$$

for all $v \in \mathbf{U}$.

Lemma 3.1. *Assume $v \in \mathbf{U}$ is such that $u_\theta^* = u^* + \theta v \in \mathbf{U}$ for $\theta \in [0, \alpha]$. Write $x_{0,t}^\theta(x_0)$ for the trajectory associated with u_θ^*. Then $z_t = \partial x_{0,t}^\theta(x_0)/\partial\theta|_{\theta=0}$ exists a.s. and is the unique solution of the equation*

$$z_t = \int_0^t (f_x(r, x_{0,r}^*(x_0), u_r^*)z_r + f_u(r, x_{0,r}^*(x_0), u_r^*)v_r)\, dr$$
$$+ \sum_{i=1}^n \int_0^t (g_x^{(i)}(r, x_{0,r}^*(x_0), u_r^*)z_r + g_u^{(i)}(r, x_{0,r}^*(x_0), u_r^*)v_r)\, dw_r^i. \qquad (3.2)$$

Proof.

$$x_{0,t}^\theta(x_0) = x_0 + \int_0^t f(r, x_{0,r}^\theta(x_0), u_r^* + \theta_{v_r})\, dr$$
$$+ \sum_{i=1}^n \int_0^t g^{(i)}(r, x_{0,r}^\theta(x_0), u^* + \theta v_r)\, dw_r^i$$

and the result follows from the theorem of Blagovescenskii and Freidlin [3] on the differentiability of solutions of stochastic partial differential equations which depend on a parameter. □

Comparing (3.2) for z and (2.3) for $D_{0,t}^* = \partial x_{0,t}^*/\partial x$, we have the following result by variation of constants:

Lemma 3.2. *Write*

$$\eta_{0,t} = \int_0^t (D_{0,r}^*)^{-1} f_v(r)v_r\, dr + \sum_{i=1}^n \int_0^t (D_{0,r}^*)^{-1} g_v^{(i)}(r)v_r\, dw_r^i$$
$$- \sum_{i=1}^n \int_0^t (D^*{}_{0r})^{-1} g_x^{(i)}(r) g_v^{(i)}(r) v_r\, dr. \qquad (3.3)$$

Then $z_t = D_{0,t}^* \eta_{0,t}$.

Proof. By differentiating, we see the product $D_{0,t}^* \eta_{0,t}$ satisfies (3.2). □

Lemma 3.3.

$$\left.\frac{dJ(u_\theta^*)}{d\theta}\right|_{\theta=0} = E\Big[c_x(x_{0,T}^*(x_0)) D_{0,T} \eta_{0,T}$$
$$+ \int_0^T (h_x(r, x_{0,r}^*(x_0), u_r^*) D_{0,r}^* \eta_{0,r} + h_v(r, x_{0,r}^*(x_0), u_r^*)v_r)\, dr \Big]. \qquad (3.4)$$

Proof

$$J(u_\theta^*) = E\left[c(x_{0,T}^\theta(x_0)) + \int_0^T h(r, x_{0,r}^\theta(x_0), u_\theta^*(r))\, dr \right].$$

The result of Lemma 3.1 and [3] justifies the differentiation in θ giving

$$\left.\frac{dJ(u_\theta^*)}{d\theta}\right|_{\theta=0} = E\left[c_x(x_{0,T}^*(x_0))z_T + \int_0^T (h_x(r, x_{0,r}^*, u_r^*)z_r + h_v(r, x_{0,r}^*, u_r^*)v_r)\, dr\right].$$

Substituting $z_t = D_{0,t}^*\eta_{0,t}$ (3.4) follows. □

Notation 3.4. Consider the right continuous version of the square integrable martingale

$$M_t := E\left[c_x(x_{0,T}^*(x_0))D_{0,T}^* + \int_0^T h_x(r, x_{0,r}^*(x_0), u_r^*)D_{0,r}^*\, dr | F_t\right].$$

It is known (see, for example, [8]) that M_t has a representation as a stochastic integral

$$M_t = E\left[c_x(x_{0,T}^*(x_0))D_{0,T}^* + \int_0^T h_x(r)D_{0,r}^*\, dr\right] + \sum_{i=1}^n \int_s^t \gamma_r^i\, dw_r^i, \tag{3.5}$$

where the γ^i are predictable processes such that

$$E\left[\int_0^T (\gamma_r^i)^2\, dr\right] < \infty.$$

We determine the γ^i below, but first we introduce some more notation. Write

$$\xi_t = M_t - \int_s^t h_x(r)D_{0,r}^*\, dr.$$

Definition 3.5. The adjoint variable is the process defined by

$$p_s = \xi_s(D_{0,s}^*)^{-1}.$$

Theorem 3.6.

$$\left.\frac{dJ(u_\theta^*)}{d\theta}\right|_{\theta=0} = E\left[\int_0^T p_s f_v(s)v_s\, ds - \sum_{i=1}^n \int_0^T p_s g_x^{(i)}(s)g_v^{(i)}(s)v_s\, ds + \int_0^T h_v(s)v_s\, ds\right.$$
$$\left. + \sum_{i=1}^n \int_0^T \gamma_s^i(D_{0,s}^*)^{-1}g_v^{(i)}(s)v_s\, ds\right]. \tag{3.6}$$

Proof. First note that

$$\begin{aligned}
\xi_T\eta_{0,T} = &\int_0^T \xi_s(D_{0,s}^*)^{-1}f_v(s)v_s\, ds \\
&+ \sum_{i=1}^n \int_0^T \xi_s(D_{0,s}^*)^{-1}g_v^{(i)}v_s\, dw_s^i - \sum_{i=1}^n \int_0^T \xi_s(D_{0,s}^*)^{-1}g_x^{(i)}(s)g_v^{(i)}(s)v_s\, ds \\
&+ \sum_{i=1}^n \int_0^T \gamma_s^i\eta_{0,s}\, dw_s^i - \int_0^T h_x(s)D_{0,s}^*\eta_{0,s}\, ds \\
&+ \sum_{i=1}^n \int_0^T \gamma_s^i(D_{0,s}^*)^{-1}\, g_v^{(i)}(s)v_s\, ds.
\end{aligned} \tag{3.7}$$

Also

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = E\left[\xi_T\eta_{0,T} + \int_0^T h_x(s)D_{0,s}^*\eta_{0,s}\,ds + \int_0^T h_v(s)v_s\,ds\right]. \tag{3.8}$$

Substituting (3.7) in (3.8) and using the definition of p_s, the result follows. □

4. Martingale Representation

Under certain conditions the minimum cost attainable under the stochastic open-loop controls is equal to the minimum cost attainable under Markov feedback controls of the form $u(s, \xi_{0,s}^u(x_0))$. See, for example, [4] and [13]. If u_M is a Markov control, with a corresponding, possibly weak, solution trajectory ξ^{u_M}, then u_M can be considered as a stochastic open-loop control $u_M(\omega)$ by setting

$$u_M(\omega) = u_M(s, \xi_{0,s}^{u_M}(x_0, \omega)).$$

This means the control in effect "follows" its original trajectory ξ^{u_M} rather than any new trajectory. The control u_M is, therefore, similar to the adjoint strategies introduced by Krylov. The point of considering the open-loop control u_M is that when we consider variations in the state trajectory ξ, and derivatives of the map $x \to \xi_{s,t}(x)$, the control does not react, and so we do not introduce derivatives in the u variable.

We assume in this section that the optimal stochastic open-loop control u^* is Markov. We can then determine the integrands γ^i in (3.5).

Lemma 4.1. *For $1 \le i \le n$*

$$\gamma_s^i = \left(\frac{\partial p_s}{\partial x}\cdot g_s^i + p_s\, g_x^{(i)}(s)\right)D_{0,s}^*.$$

Proof. Consider a stochastic system with components $x_{0,t}^*(x_0)$, $D_{0,t}^*$, and $Y_{0,t}$, where

$$y_{0,t} = \int_0^t h_x(r, x_{0,r}^*(x_0), u_r^*)D_{0,r}^*\,dr.$$

Now

$$p_t D_{0,t}^* + y_{0,t} = M_t = E\left[c_x(x_{0,T}^*(x_0))D_{0,T}^* + \int_0^T h_x(r)D_{0,r}^*\,dr \,|\, F_t\right].$$

Writing $x = x_{0,t}^*(x_0)$, $D = D_{0,t}^*$, $y = y_{0,t}$ by the Markov property this expectation is the same as

$$E\left[c_x(x_{t,T}^*(x))D_{t,T}D + \int_t^T h_x(r)D_{t,r}^*\,dr D + y \,|\, x, y, D\right] = V(t, x, y, D).$$

From the left-hand side this also equals $p_t(x)D + y$. The martingale representation result of Elliott and Kohlmann [9], [10] follows from the Ito formula, using the

differentiability of the solutions of stochastic differential equations as functions of their initial conditions, and equating the bounded variation terms to zero. Therefore,

$$M_t = E\left[c_x D + \int_0^T h_x D\, dr\right] + \sum_{i=1}^{n} \int_0^t \gamma_r^i\, dw_r^i$$

$$= V(0, x_0, 0, I) + \sum_{i=1}^{n} \int_0^t p_x(r) g^{(i)}(r) D_{0,r}^*\, dw_r^i + \sum_{i=1}^{n} \int_0^t p(r) g_x^{(i)}(r) D_{0,r}^*\, dw_r^i. \tag{4.1}$$

That is, $\gamma_r^i = (p_x(r) g^{(i)}(r) + p(r) g_x^{(i)}(r)) D_{0,r}^*$. □

Remarks 4.2. Substituting in (3.6) we have

$$\left.\frac{dJ(u_\theta^*)}{d\theta}\right|_{\theta=0} = E\left[\int_0^T p_s f_v(s) v_s\, ds + \sum_{i=1}^{n} \int_0^T p_x(s) g^{(i)}(s) g_v^{(i)}(s) v_s\, ds + \int_0^T h_v(s) v_s\, ds\right].$$

Returning to the perturbation

$$u_\theta(t) = u^*(t) + \theta(v(t) - u^*(t))$$

of the optimal control, we have

$$\left.\frac{dJ(u_\theta)}{d\theta}\right|_{\theta=0} \geq 0.$$

That is,

$$E\left[\int_0^T p_s f_v(s)(v_s - u_s^*)\, ds + \sum_{i=1}^{n} \int_0^T p_x(s) g^{(i)}(s) g_v^{(i)}(s)(v_s - u_s^*)\, ds + \int_0^T h_v(s)(v_s - u_s^*)\, ds\right] \geq 0 \tag{4.2}$$

for all $v \in \mathrm{U}$. Define the Hamiltonian by

$$H(x, v, t, p(t)) = p(t) f(t, x, v) + \sum_{i=1}^{n} p_x(t) g^{(i)}(t, x, u_t^*) g^{(i)}(t, x, v) + h(t, x, v). \tag{4.3}$$

Then, because (4.2) is true for all $v \in \mathrm{U}$, we have the following result:

Theorem 4.3. *If u^* is the optimal control, then a.e. t and a.s. w*

$$\frac{\partial H}{\partial v}(x_{0,t}^*(x_0), u_t^*, t, p(t)) \cdot (v_t - u_t^*) \geq 0$$

for all $v \in \mathrm{U}$.

Finally we derive the equations satisfied by p. We first show p satisfies a forward stochastic partial differential equation.

Theorem 4.4.

$$
\begin{aligned}
p_t = {} & E\left[c_x(x^*_{0,T}(x_0))D^*_{0,T} + \int_s^t h_x(r, x^*_{0,r}(x_0), u^*_r)D^*_{0,r}\,dr\right] \\
& - \int_0^t p_r f_x(r, x^*_{0,r}(x_0), u^*_r)\,dr - \int_0^t h_x(r, x^*_{0,r}(x_0), u^*_r)\,dr \\
& + \int_0^t p_x(r)g(r, x^*_{0,r}(x_0), u^*_r)\,dw_r \\
& - \sum_{i=1}^n \int_0^t p_x(r)g^{(i)}(r, x^*_{0,r}(x_0), u^*_r)g^{(i)}_x(r, x^*_{0,r}(x_0), u^*_r)\,dr.
\end{aligned}
\tag{4.4}
$$

Proof. From (4.1)

$$
\begin{aligned}
M_t &= p_t D^*_{0,t} + y_{0,t} \\
&= E\left[c_x(x^*_{0,T(x_0)})D^*_{0,T} + \int_0^T h_x(r)D^*_{0,r}\,dr\right] \\
&\quad + \sum_{i=1}^n \int_0^t p_x(r)g^{(i)}(r)D^*_{0,r}\,dw^i_r + \sum_{i=1}^n \int_0^t p(r)g^{(i)}_x(r)D^*_{0,r}\,dw^i_r.
\end{aligned}
$$

Multiplying by $H^*_{0,t} = (D^*_{0,t})^{-1}$, whose equation is given by (2.4), we see

$$
\begin{aligned}
p_t &= (p_t D^*_{0,t})H^*_{0,t} \\
&= p_0 - \int_0^t p_r\left(f_x(r, x^*_{0,r}(x), u^*_r) - \sum_{i=1}^n g^{(i)}_x(r, x^*_{0,r}(x), u^*_r)^2\right)dr \\
&\quad - \sum_{i=1}^n \int_0^t p_r g^{(i)}_x(r, x^*_{0,r}(x), u^*_r)\,dw^i_r - \int_0^t h_x(r, x^*_{0,r}(x), u^*_r)\,dr \\
&\quad + \sum_{i=1}^n \int_0^t p_x(r)g^{(i)}(r)\,dw^i_r + \sum_{i=1}^n \int_0^t p_r g^{(i)}_x(r)\,dw^i_r \\
&\quad - \sum_{i=1}^n \int_0^t p_x(r)g^{(i)}(r)g^{(i)}_x(r)\,dr - \sum_{i=1}^n \int_0^t p_r g^{(i)}_x(r)g^{(i)}_x(r)\,dr \\
&= p_0 - \int_0^t p_r f_x(r)\,dr - \int_0^t h_x(r)\,dr \\
&\quad + \int_0^t p_x(r)g(r)\,dw_r - \sum_{i=1}^n \int_0^t p_x(r)g^{(i)}(r)g^{(i)}_x(r)\,dr.
\end{aligned}
$$

Here

$$p_0 = E\left[c_x(x^*_{0,T}(x_0))D^*_{0,T} + \int_0^t h_x(r, x^*_{0,r}(x_0), u^*_r)D^*_{0,r}\,dr\right]$$

and the functions $f_x(r)$, $h_x(r)$, $g(r)$, $g_x^{(i)}(r)$ are evaluated at the argument

$(r, x^*_{0,r}(x_0), u^*_r)$. □

Remarks 4.5. Although (4.4) is a forward stochastic partial differential equation for p it does not appear to have a unique solution; certainly giving any (constant) initial condition does not determine a corresponding vector function solution p. We now show p is given by a backward, nonstochastic, parabolic system of partial differential equations which, under the given conditions, has a unique solution.

Theorem 4.6. *p is the solution of the backward parabolic system of partial differential equations*

$$\frac{\partial p}{\partial t} + p_x(t)f(t) + h_x(t) + p(t)f_x(t) + \tfrac{1}{2}\sum_{i=1}^{n} p_{xx}(t)g^{(i)}(t)\otimes g^{(i)}(t) = 0$$

with terminal condition

$$p_T = c_x(x).$$

Proof. Consider again the function

$$M_t = V(t, x, y, D) = p_t(x)D + y$$

of Lemma 4.1. Writing again the Ito formula we have

$$V(t, x, y, D) = V(0, x_0, 0, I) + \int_0^t \frac{\partial V}{\partial r} + \frac{\partial V}{\partial x}\cdot f(r) + \frac{\partial V}{\partial y}\cdot h_x(r)D^*_{0,r}$$

$$+ \frac{\partial V}{\partial D}\cdot f_x(r)D^*_{0,r} + \tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 V}{\partial x^2}g^{(i)}(r)\otimes g^{(i)}(r)\,dr$$

$$+ \text{the two stochastic integral terms in (4.1).} \tag{4.5}$$

However, $M_t = V(t, x, y, D)$ is a martingale so, as observed in the proof of Lemma 4.1, the dr integral (that is, the bounded variation term) must be identically zero. Therefore, the dr integrand in (4.5) must be identically zero, so

$$\frac{\partial V}{\partial r} + \frac{\partial V}{\partial x}\cdot f(r) + \frac{\partial V}{\partial y}\cdot h_x(r)D^*_{0,r} + \frac{\partial V}{\partial D}\cdot f_x(r)D^*_{0,r} + \tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 V}{\partial x^2}g^{(i)}(r)\otimes g^{(i)}(r) = 0.$$

Recall $V = p_t(x)D + y$, so

$$\left(\frac{\partial p}{\partial t} + p_x(t)f(t) + h_x(t) + p(t)f_x(t) + \tfrac{1}{2}\sum_{i=1}^{n} p_{xx}(t)g^{(i)}(t)\otimes g^{(i)}(t)\right)D^*_{0,t} = 0.$$

$D^*_{0,t}$ is nonsingular, so we see that p is the solution of

$$\frac{\partial p}{\partial t} + p_x(t)f(t) + h_x(t + p(t)f_x(t) + \tfrac{1}{2}\sum_{i=1}^{n} p_{xx}(t)g^{(i)}(t)\otimes g^{(i)}(t) = 0$$

with terminal condition

$$p_T = c_x(x). \qquad \square$$

5. Conclusion

Using the differentiability result of [3] the proof of Bensoussan [2] is simplified and the adjoint process, when the control appears in the diffusion term, is obtained. Furthermore, by applying the martingale representation result of Elliott and Kohlmann [9], [10], explicit equations for the adjoint process are established.

Under certain conditions (see [5] and [15]), if the optimal control is Markov we can write

$$\begin{aligned} V(s, x) &= E\left[c(x^*_{0,T}(x_0)) + \int_t^T h(r, x^*_{0,r}(x_0), u^*_r)\,dr \,\middle|\, F_t\right] \\ &= E_{s,x}\left[c(x^*_{t,T}(x)) + \int_t^T h(r, x^*_{t,r}(x), u^*_r)\,dr\right], \end{aligned}$$

where $x = x^*_{0,t}(x_0)$, for the optimum remaining cost. Then, at least formally, we see $p_s(x)$ is the gradient $V_x(s, x)$; so p_x is $V_{xx}(s, x)$.

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MARTINGALES ASSOCIATED WITH FINITE MARKOV CHAINS

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1. Introduction.

In a recent paper, [1], Phillipe Biane introduced martingales M^k associated with the different jump 'sizes' of a time homogeneous, finite Markov chain and developed homogeneous chaos expansions. It has long been known that the Kolmogorov equation for the probability densities of a Markov chain gives rise to a canonical martingale M. The modest contributions of this note, are that working with a non-homogeneous chain, we relate Biane's martingales M^k to M, calculate the quadratic variation of M and thereby that of the M^k. In addition, square field identities are obtained for each jump size.

For $0 \le i \le N$ write $e_i = (0, 0, \ldots, 1, \ldots, 0)^*$ for the i-th unit (column) vector in R^{N+1}, (so $e_0 = (1, 0, \ldots, 0)^*$ etc.). Consider the (non-homogeneous) Markov process $\{X_t\}$, $t \ge 0$, defined on a probability space (Ω, F, P), whose state space, without loss of generality, can be identified with the set $S = \{e_0, e_1, \ldots, e_N\}$. Write $p_t^i = P(X_t = e_i)$, $0 \le i \le N$. We shall suppose that for some family of matrices A_t, $p_t = (p_t^0, \ldots, p_t^N)^*$ satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = A_t p_t. \tag{1.1}$$

$A_t = (a_{ij}(t))$ is, therefore, the family of Q-matrices of the process.

It has long been known (see, for example, Liptser and Shiryayev [4], Elliott [2]) that the process

$$M_t = X_t - X_0 - \int_0^t A_r X_{r-} dr \tag{1.2}$$

is a martingale. (See Lemma 2.3 below.)

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Solving (1.2) by 'variation of constants' we can immediately write

$$X_t = \Phi(t,0)\Big(X_0 + \int_0^t \Phi(0,r)^{-1} dM_r\Big) \tag{1.3}$$

where Φ is the fundamental matrix of the generator A. Equation (1.3) is a martingale representation result which in turn gives a representation result in terms of the M^k. (By iterating this representation Biane's homogeneous chaos expansion can be obtained; this is quite explicit, in terms of matrices Φ and matrices associated with A.) Functions of the chain are just given by vectors in R^{N+1} and in Section 4 'square field' identities are obtained for each jump 'size'.

2. Markov Chains.

Consider a Markov chain $\{X_t\}$, $t \geq 0$, with state space $S = \{e_0, \dots, e_N\}$ and Q-matrix generators A_t. We shall make the following assumptions.

ASSUMPTIONS 2.1. (i) For all $0 \leq i,j \leq N$ and $t \geq 0$

$$|a_{ij}(t)| \leq B' \tag{2.1}$$

for some bound B'; write $B = B' + 1$.

(ii) For all $0 \leq i,j \leq N$ and $t \geq 0$, $a_{ij}(t) > 0$ if $i \neq j$ and, (because A_t is a Q-matrix),

$$a_{ii}(t) = -\sum_{j \neq i} a_{ji}(t). \tag{2.2}$$

The fundamental transition matrix associated with A will be denoted by $\Phi(t,s)$, so with I the $(N+1) \times (N+1)$ identity matrix,

$$\frac{d\Phi(t,s)}{dt} = A_t \Phi(t,s), \qquad \Phi(s,s) = I \tag{2.3}$$

and

$$\frac{d\Phi(t,s)}{ds} = -\Phi(t,s) A_s, \qquad \Phi(t,t) = I. \tag{2.4}$$

(If A_t is constant $\Phi(t,s) = \exp A(t-s)$.)

BOUNDS 2.2. For a matrix $C = (c_{ij})$ consider a norm $|C| = \max_{i,j} |c_{ij}|$. Then for all t, $|A_t| \leq B$. The columns of Φ are probability distributions so $|\Phi(t,s)| \leq 1$ for all t, s.

Consider the process in state $x \in S$ at time s and write $X_{s,t}(x)$ for its state at time $t \geq s$.

Then $E[X_{s,t}(x)] = E_{s,x}[X_t] = \Phi(t,s)x$. Write F_t^s for the right continuous complete filtration generated by $\sigma\{X_r : s \leq r \leq t\}$ and $F_t^0 = F_t$.

LEMMA 2.3. *The process* $M_t = X_t - X_0 - \int_0^t A_r X_{r-} dr$ *is an* $\{F_t\}$ *martingale.*

Proof. Suppose $0 \le s \le t$. Then

$$\begin{aligned} E[M_t - M_s \mid F_s] &= E\Big[X_t - X_s - \int_s^t A_r X_{r-} dr \mid F_s\Big] \\ &= E\Big[X_t - X_s - \int_s^t A_r X_r dr \mid X_s\Big] \\ &= E_{s,X_s}[X_t] - X_s - \int_s^t A_r E_{s,X_s}[X_r] dr \\ &= \Phi(t,s)X_s - X_s - \int_s^t A_r \Phi(r,s) X_s dr = 0 \qquad \text{by (2.3).} \end{aligned}$$

Therefore,

$$X_t = X_0 + \int_0^t A_r X_r dr + M_t = X_0 + \int_0^t A_r X_{r-} dr + M_t$$

where M is an $\{F_t\}$ martingale.

NOTATION 2.4. If $x = (x_0, x_1, \dots, x_N)^* \in R^{N+1}$ then diag x is the matrix

$$\begin{pmatrix} x_0 & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_N \end{pmatrix}.$$

LEMMA 2.5.

$$\langle M, M \rangle_t = \text{diag} \int_0^t A_r X_{r-} dr - \int_0^t (\text{diag } X_{r-}) A_r^* dr - \int_0^t A_r (\text{diag } X_{r-}) dr.$$

Proof. Recall $X_t \in S$ is one of the unit vectors e_i. Therefore,

$$X_t \otimes X_t = \text{diag } X_t. \tag{2.5}$$

Now by the differentiation rule

$$\begin{aligned} X_t \otimes X_t = X_0 \otimes X_0 &+ \int_0^t X_{r-} \otimes (A_r X_{r-}) dr \\ &+ \int_0^t X_{r-} \otimes dM_r + \int_0^t (A_r X_{r-}) \otimes X_{r-} dr \\ &+ \int_0^t dM_r \otimes X_{r-} + \langle M, M \rangle_t + N_t \end{aligned}$$

where N_t is the F_t martingale

$$[M,M]_t - \langle M,M\rangle_t.$$

However, a simple calculation shows

$$X_{r-} \otimes (A_r X_{r-}) = (\text{diag } X_{r-})A_r^*$$

and

$$(A_r X_{r-}) \otimes X_{r-} = A_r(\text{diag } X_{r-}).$$

Therefore,

$$X_t \otimes X_t = X_0 \otimes X_0 + \int_0^t (\text{diag } X_{r-})A_r^* dr$$

$$+ \int_0^t A_r(\text{diag } X_{r-})dr + \langle M,M\rangle_t + \text{ martingale.} \tag{2.6}$$

Also, from (2.5)

$$X_t \otimes X_t = \text{diag } X_t = \text{diag } X_0 + \text{diag } \int_0^t A_r X_{r-} dr + \text{diag } M_t. \tag{2.7}$$

The semimartingale decompositions (2.6) and (2.7) must be the same, so equating the predictable terms

$$\langle M,M\rangle_t = \text{diag } \int_0^t A_r X_{r-} dr - \int_0^t (\text{diag } X_{r-})A_r^* dr - \int_0^t A_r(\text{diag } X_{r-})dr.$$

We next note the following representation result:

LEMMA 2.6.
$$X_t = \Phi(t,0)\Big(X_0 + \int_0^t \Phi(r,0)^{-1} dM_r\Big). \tag{2.8}$$

Proof. This result follows immediately by 'variation of constants'.

REMARKS 2.7. A function of $X_t \in S$ can be represented by a vector

$$f(t) = (f_0(t), \ldots, f_N(t))^* \in R^{N+1}$$

so that $f(t, X_t) = f(t)^* X_t = \langle f(t), X_t\rangle$ where $\langle\ ,\ \rangle$ denotes the inner product in R^{N+1}.

We, therefore, have the following differentiation rule and representation result:

LEMMA 2.8. *Suppose the components of $f(t)$ are differentiable in t. Then*

$$f(t, X_t) = f(0, X_0) + \int_0^t \langle f'(r), X_r \rangle dr + \int_0^t \langle f(r), A_r X_{r-} \rangle dr + \int_0^t \langle f(r), dM_r \rangle. \tag{2.9}$$

Here, $\int_0^t \langle f(r), dM_r \rangle$ is an F_t-martingale. Also,

$$f(t, X_t) = \langle f(t), \Phi(t,0) X_0 \rangle + \int_0^t \langle f(t), \Phi(t,r) dM_r \rangle. \tag{2.10}$$

This gives the martingale representation of $f(t, X_t)$.

REMARK 2.9. With an obvious abuse of notation, if the jump times of the chain are $T_1(w), T_2(w), \ldots$, we can write down a 'random measure' decomposition of X_t from (1.2) as

$$X_t = X_0 + \int_0^t \sum_i (e_i - X_{r-})(\sum_k \delta_{T_k(w)}(dr) \delta_{i_k(w)}(i) - a_{iX_{r-}} dr)$$
$$+ \int_0^t \sum_i (e_i - X_{r-}) a_{iX_{r-}} dr,$$

because $\sum_i (e_i - X_{r-}) a_{iX_{r-}} = A_{r-} X_{r-}$. Here, $\delta_{T_k(w)}(dr)$ is the unit mass at $T_k(w)$ and, with $X_{T_k(w)} = e_{i_k(w)}$, $\delta_{i_k(w)}(i)$ is 1 if $i = i_k(w)$ and 0 otherwise. That is,

$$M_t = \int_0^t \sum_i (e_i - X_{r-})(\sum_k \delta_{T_k(w)}(dr) \delta_{i_k(w)}(i) - a_{iX_{r-}} dr).$$

This representation would provide another means of calculating $\langle M, M \rangle_t$.

3. Shift Operators.

The formulae of Section 2, particularly the martingale representations (2.8) and (2.10), provide basic information about the Markov process X. However, if the 'size' of the jumps is considered some other expressions, including a homogeneous chaos expansion, were obtained recently by Biane [1]. We wish to indicate how the results of Biane relate to the above expressions. First we introduce some notation.

NOTATION 3.1. Write $i \oplus j$ for addition $\mod (N+1)$. For $X_s \in S = \{e_0, e_1, \ldots, e_N\}$, say $X_s = e_i$, and $k = 1, \ldots, N$, write

$$X_s^k = e_{i \oplus k}.$$

That is, $X_s \to X_s^k$ corresponds to a cyclic jump of size k in the index of the unit vector corresponding to the state.

Suppose $X_{s-} = e_i$ and $X_{s-}^k = e_j$, where $j = i \oplus k$, then clearly

$$(X_{s-}^k)^* A_s X_{s-} = a_{ji}(s). \tag{3.1}$$

We now wish to introduce some subsidiary matrices associated with $A_s = (a_{ij}(s))$. These can best be explained by first considering the 3×3 case. Suppose

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{21} \end{pmatrix}.$$

Then

$$A^1 := \begin{pmatrix} -a_{10} & 0 & a_{02} \\ a_{10} & -a_{21} & 0 \\ 0 & a_{21} & -a_{02} \end{pmatrix},$$

$$A^2 := \begin{pmatrix} -a_{20} & a_{01} & 0 \\ 0 & -a_{01} & a_{12} \\ a_{20} & 0 & -a_{12} \end{pmatrix}.$$

Note that if A is a Q-matrix $a_{0i} + a_{1i} + a_{2i} = 0$, so $A^1 + A^2 = A$.

In general, if $A_s = (a_{ij}(s))$ is an $(N+1) \times (N+1)$ Q-matrix, A_s^k is obtained by forming a matrix from the k-th subdiagonal (continued as a superdiagonal), with the negative of the column entries on the diagonal and zeros elsewhere. By construction, A^k is a Q-matrix, and it is clearly related to those jumps of 'size' k. As above,

$$A_s = \sum_{k=1}^{N} A_s^k. \tag{3.2}$$

Also,

$$((X_{s-}^k)^* A_s X_{s-})(X_{s-}^k - X_{s-}) = A_s^k X_{s-}, \tag{3.3}$$

so

$$\sum_{k=1}^{N} ((X_{s-}^k)^* A_s X_{s-})(X_{s-}^k - X_{s-}) = A_s X_{s-}. \tag{3.4}$$

We also wish to introduce matrices $\widetilde{A}^k$, $k \neq 0$, whose off-diagonal entries are the (positive) square roots of those of A^k, and whose diagonal entries are the

negative of that square root in the same column. That is, in the (3×3) case above:

$$\widetilde{A}^1 := \begin{pmatrix} -\sqrt{a_{10}} & 0 & \sqrt{a_{02}} \\ \sqrt{a_{10}} & -\sqrt{a_{21}} & 0 \\ 0 & \sqrt{a_{21}} & -\sqrt{a_{02}} \end{pmatrix}$$

$$\widetilde{A}^2 := \begin{pmatrix} -\sqrt{a_{20}} & \sqrt{a_{01}} & 0 \\ 0 & -\sqrt{a_{01}} & \sqrt{a_{12}} \\ \sqrt{a_{20}} & 0 & -\sqrt{a_{12}} \end{pmatrix}.$$

For $k = 1, \ldots, N$ write

$$\Lambda_s^k = ((X_{s-}^k)^* A_s X_{s-})^{-1/2} (X_{s-}^k)^*,$$

so Λ_s^k is a predictable process.

DEFINITION 3.2. In our notation the matrices M^k introduced by Biane [1] are, for $k = 1, \ldots, N$

$$M_t^k = \sum_{0 \le s \le t} ((X_{s-}^k)^* A_s X_{s-})^{-1/2} I(X_s = X_{s-}^k) - \int_0^t ((X_{s-}^k)^* A_s X_{s-})^{1/2} ds. \tag{3.5}$$

LEMMA 3.3. *For $k = 1, \ldots, N$,*

$$M_t^k = \int_0^t \Lambda_s^k \cdot dM_s.$$

Proof. First note

$$(X_{s-}^k)^* \cdot dX_s = (X_{s-}^k)^* \cdot \Delta X_s = (X_{s-}^k)^* \cdot (X_s - X_{s-}) = I(X_s = X_{s-}^k). \tag{3.6}$$

Also, $X_t = X_0 + \int_0^t A_s X_{s-} ds + M_t$, so

$$\begin{aligned} M_t^k &= \int_0^t \Lambda_s^k \cdot dX_s - \int_0^t \Lambda_s^k \cdot A_s X_{s-} ds \\ &= \int_0^t ((X_{s-}^k)^* A_s X_{s-})^{-1/2} (X_{s-}^k)^* \cdot dX_s \\ &\quad - \int_0^t ((X_{s-}^k)^* A_s X_{s-})^{-1/2} ((X_{s-}^k)^* A_s X_{s-}) ds, \end{aligned}$$

and the result follows from (3.6).

LEMMA 3.4. *For $k = 1, \ldots, N$, $\langle M^k, M^k \rangle_t = t$.*

Proof. $M_t^k = \int_0^t \Lambda_s^k \cdot dM_s$, so

$$\langle M^k, M^k \rangle_t = \int_0^t \Lambda_s^k d\langle M, M \rangle_s (\Lambda_s^k)^*$$

$$= \int_0^t \big((X_{s-}^k)^* A_s X_{s-}\big)^{-1/2} (X_{s-}^k)^*$$

$$\cdot \big(\text{diag }(A_s X_{s-}) - (\text{diag } X_{s-}) A_s^* - A_s(\text{diag } X_{s-})\big)$$

$$\cdot (X_{s-}^k)\big((X_{s-}^k)^* A_s X_{s-}\big)^{-1/2} ds.$$

Now for $k \neq 0$:

$$(X_{s-}^k)^* \text{diag } X_{s-} = 0 = (\text{diag } X_{s-})(X_{s-}^k)$$

and

$$(X_{s-}^k)^* \cdot (\text{diag }(A_s X_{s-})) \cdot (X_{s-}^k) = (X_{s-}^k)^* A_s X_{s-}.$$

Therefore, $\langle M^k, M^k \rangle_t = \int_0^t ds = t.$ □

REMARKS 3.5. For $k \neq \ell$, M^k and M^ℓ have no common jumps, so $[M^k, M^\ell]_t = 0$ and $\langle M^k, M^\ell \rangle_t = 0$. Therefore, $M^1, \ldots, M^N$ are a family of orthogonal martingales, each of which has predictable variation t.

Having expressed M^k in terms of M we now wish to express M in terms of the M^k.

THEOREM 3.6. *$M_t = \sum_{k=1}^N \int_0^t \widetilde{A}_s^k X_{s-} dM_s^k$, so the M^k form a basis.*

Proof. From (3.6) first note that

$$dX_s = \sum_{k=1}^N (X_{s-}^k - X_{s-})(X_{s-}^k)^* \cdot dX_s.$$

Therefore,

$$X_t - X_0 = \int_0^t A_s X_{s-} ds + M_t = \int_0^t dX_s \tag{3.7}$$

$$= \sum_{k=1}^N \int_0^t (X_{s-}^k - X_{s-})(X_{s-}^k)^* \cdot (A_s X_{s-} ds + dM_s). \tag{3.8}$$

By definition $dM_s^k = ((X_{s-}^k)^* A_s X_{s-})^{-1/2}(X_{s-}^k)^* \cdot dM_s$ so $(X_{s-}^k)^* \cdot dM_s = ((X_{s-}^k)^* A_s X_{s-})^{1/2} dM_s^k$. Substituting in (3.8)

$$X_t - X_0 = \int_0^t \sum_{k=1}^N (X_{s-}^k - X_{s-})((X_{s-}^k)^* A_s X_{s-}) ds$$

$$+ \sum_{k=1}^N \int_0^t (X_{s-}^k - X_{s-})((X_{s-}^k)^* A_s X_{s-})^{1/2} dM_s^k.$$

From (3.3) and (3.4) this equals

$$= \int_0^t A_s X_{s-} ds + \sum_{k=1}^N \int_0^t \widetilde{A}_s^k X_{s-} dM_s^k. \tag{3.9}$$

Comparing (3.7) and (3.9) we see

$$M_t = \sum_{k=1}^N \int_0^t \widetilde{A}_s^k X_{s-} dM_s^k. \tag{3.10}$$

4. Discrete Derivatives for Different Jump Sizes.

Consider a function f on $S = \{e_i\}$. For simplicity we suppose f is constant in time. Then, as noted in Section 2, f is represented by a vector $f = (f_0, \dots, f_N)^*$ and

$$f(X_t) = \langle f, X_t \rangle = \langle f, X_0 \rangle + \int_0^t \langle f, A_r X_{r-} \rangle dr + \langle f, M_t \rangle$$

$$= \langle f, X_0 \rangle + \int_0^t \langle f, A_r X_{r-} \rangle dr + \sum_{k=1}^N \int_0^t \langle f, \widetilde{A}_r^k X_{r-} \rangle dM_r^k. \tag{4.1}$$

from (3.9), and this is

$$= \langle f, X_0 \rangle + \int_0^t \langle A_r^* f, X_{r-} \rangle dr + \sum_{k=1}^N \int_0^t \langle (\widetilde{A}_r^k)^* f, X_{r-} \rangle dM_r^k. \tag{4.2}$$

We now re-establish the 'square field' formula of Biane [1] by calculating $f(X_t)^2$ in two ways.

LEMMA 4.1. $A_r^* f^2 - 2f \cdot A_r^* f = \sum_{k=1}^N ((\widetilde{A}_r^k)^* f)^2$.

Proof. Function multiplication is pointwise in each coordinate, so f^2 corresponds to the vector $(f_0^2, \dots, f_N^2)^*$, and

$$f^2(X_t) = \langle f^2, X_0 \rangle + \int_0^t \langle A_r^*, f^2, X_{r-} \rangle dr + \sum_{k=1}^N \int_0^t \langle (\widetilde{A}_r^k)^* f^2, X_{r-} \rangle dM_r^k \tag{4.3}$$

$$= (f(X_t))^2.$$

Using the differentiation rule this also equals

$$= f(X_0)^2 + 2\int_0^t f(X_{r-})df(X_r) + [f(X), f(X)]_t$$

$$= f(X_0)^2 + 2\int_0^t \langle f, X_{r-}\rangle\langle A_r^* f, X_{r-}\rangle dr$$

$$+ 2\sum_{k=1}^N \int_0^t \langle f, X_{r-}\rangle\langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle dM_r^k + [f(X), f(X)]_t. \tag{4.4}$$

Now

$$[f(X), f(X)]_t = \sum_{0\le r\le t} \Delta f(X_r)\Delta f(X_r)$$

$$= \sum_{k=1}^N \sum_{0\le r\le t} \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2(\Delta M_r^k)^2$$

$$= \sum_{k=1}^N \int_0^t \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2\big((X_{r-}^k)^* A_r X_{r-}\big)^{-1/2} dM_r^k$$

$$+ \sum_{k=1}^N \int_0^t \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2 dr, \qquad \text{from (3.5).}$$

Substituting in (4.4)

$$f(X_t)^2 = f(X_0)^2 + 2\int_0^t \langle f, X_{r-}\rangle\langle A_r^* f, X_{r-}\rangle dr$$

$$+ 2\sum_{k=1}^N \int_0^t \langle f, X_{r-}\rangle\langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle dM_r^k$$

$$+ \sum_{k=1}^N \int_0^t \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2\big((X_{r-}^k)^* A_r X_{r-}\big)^{-1/2} dM_r^k$$

$$+ \sum_{k=1}^N \int_0^t \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2 dr. \tag{4.5}$$

The special semimartingales (4.3) and (4.5) are equal, so equating the bounded variation terms

$$\langle A_r^* f^2, X_{r-}\rangle = 2\langle f, X_{r-}\rangle\langle A_r^* f, X_{r-}\rangle + \sum_{k=1}^N \langle(\widetilde{A}_r^k)^* f, X_{r-}\rangle^2.$$

That is, as functions on S

$$\sum_{k=1}^{N} \left((\widetilde{A}_r^k)^* f\right)^2 = A_r^* f^2 - 2f \cdot A_r^* f.$$

$\square$

$(\widetilde{A}_r^k)^*$ corresponds to a discrete derivative of 'amount', or in 'direction' k. However, the algebra suggests that $(\widetilde{A}_r^k)^2$ should be related to A_r^k.

A more specific relation is now obtained.

LEMMA 4.2. *For $k = 1, \ldots, N$*

$$\left((\widetilde{A}_r^k)^* f\right)^2 = (A_r^k)^* f^2 - 2f \cdot (A_r^k)^* f.$$

Proof. From the form of $\widetilde{A}^k$ and A^k, for any $f \in R^{N+1}$

$$\begin{aligned}
(A^k)^* f &= \big(a_{k0}(-f_0 + f_k), a_{k\oplus 1,1}(-f_1 + f_{k\oplus 1}), \\
&\qquad \ldots, a_{k\oplus N,N}(-f_N + f_{k\oplus N})\big), \\
(A^k)^* f^2 &= \big(a_{k0}(-f_0^2 + f_k^2), a_{k\oplus 1,1}(-f_1^2 + f_{k\oplus 1}^2), \\
&\qquad \ldots, a_{k\oplus N,N}(-f_N^2 + f_{k\oplus N}^2)\big), \\
(\widetilde{A}^k)^* f &= \big(\sqrt{a_{k0}}(-f_0 + f_k), \sqrt{a_{k\oplus 1,1}}(-f_1 + f_{k\oplus 1}), \\
&\qquad \ldots, \sqrt{a_{k\oplus N,N}}(-f_N + f_{k\oplus N})\big).
\end{aligned}$$

Therefore, as function multiplication is pointwise, that is coordinatewise:

$$\begin{aligned}
\left((\widetilde{A}^k)^* f\right)^2 &= \big(a_{k0}(f_0^2 - 2f_0 f_k + f_k^2), \ldots, a_{k\oplus N,N}(f_N^2 - 2f_N f_{k\in N} + f_{k\oplus N}^2)\big) \\
f((A^k)^* f) &= \big(a_{k0}(-f_0^2 + f_0 f_k), \ldots, a_{k\oplus N,N}(-f_N^2 + f_N f_{k\oplus N})\big).
\end{aligned}$$

Operating coordinatewise, for example,

$$(-f_j^2 + f_{k\oplus j}^2) - 2(-f_j^2 + f_j f_{k\oplus j}) = f_j^2 - 2f_j f_{k\oplus j} + f_{k\oplus j}^2$$

and the result follows. $\square$

Finally, we note that substituting (3.10) in (2.9) we have

$$X_t = \Phi(t,0)\Big(X_0 + \sum_{k=1}^{N} \int_0^t \Phi(r,0)^{-1} \widetilde{A}_r^k X_{r-} dM_r^k\Big). \tag{4.6}$$

Now X_{r-} is a.s. equal to X_r which equals

$$X_r = \Phi(r,0)\Big(X_0 + \sum_{k_2=1}^{N} \int_0^r \Phi(r_2,0)^{-1} \widetilde{A}_{r_2}^{k_2} X_{r_2-} dM_{r_2}^{k_2}\Big).$$

Substituting in (5.1) we have

$$X_t = \Phi(t,0)X_0 + \sum_{k=1}^{N} \int_0^t \Phi(t,r)\widetilde{A}_r^k \Phi(r,0) X_0 dM_r^k$$

$$+ \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \int_0^t \int_0^{r_1} \Phi(t,r_1)\widetilde{A}_{r_1}^{k_1} \Phi(r_1,r_2)\widetilde{A}_{r_2}^{k_2} X_{r_2-} dM_{r_2}^{k_2} dM_{r_2}^{k_1}.$$

Iterating this process we obtain the homogeneous chaos expansions of Biane [1], (see also Elliott and Kohlmann [3]), in terms of the non-homogeneous transition matrices Φ and the matrices $\widetilde{A}^k$.

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A PARTIALLY OBSERVED CONTROL PROBLEM FOR MARKOV CHAINS

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Abstract: A finite state, continuous time Markov chain is considered and the solution to the filtering problem given when the observation process counts the total number of jumps. The Zakai equation for the unnormalized conditional distribution is obtained and the control problem discussed in separated form with this as the state. A new feature is that, because of the correlation between the state and observation process, the control parameter appears in the "diffusion" coefficient which multiplies the Poisson noise in the Zakai equation. By introducing a Gateaux derivative the minimum principle, satisfied by an optimal control, is derived. If the optimal control is Markov a stochastic integrand can be obtained more explicitly and new forward and backward equations satisfied by the adjoint process are obtained.

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A Partially Observed Control Problem for Markov Chains

ROBERT J. ELLIOTT

1. Introduction.

In this paper a finite state space, continuous time Markov chain is considered. The state space of the chain is taken to be, without loss of generality, the set of unit vectors $S = \{e_i\}$, $e_i = (0, 0, \dots, 1, \dots, 0)^*$ of R^{N+1}, thus facilitating the use of linear algebra. Basic martingales associated with the Markov chain are identified and the solution to a natural filtering problem associated with the chain is given. This describes the recursive estimation of the state of the Markov chain if only the total number of jumps is observed. Such a filtering formula can be obtained by specializing results in the book of Brémaud [5]; however, using the basic martingales our notation and framework is quite simple. The related Zakai equation for the unnormalized conditional distribution is then obtained. In Section 5 the optimal control of a Markov chain is discussed in the partially observed case, when only the total number of jumps is known to the controller. This control problem is treated using the 'separation principle', by discussing the control of the unnormalized distribution given by the Zakai equation — that is, the filtering problem has been separated from the control problem. The Zakai equation is a linear, vector equation driven by a standard Poisson process. Because the observation process, which counts the number of jumps, is correlated with the state process the signal and observation processes are correlated, so, in contrast to earlier work on controlled Markov chains, the control variable occurs in the 'diffusion' coefficient of the Zakai equation, multiplying the compensated Poisson process noise. A minimum principle is obtained by adapting techniques of Bensoussan [1] and calculating a Gateaux derivative. Finally, if the optimal control is Markov the integrand in a martingale representation can be obtained more explicitly. This enables the adjoint process to be described, and new forward and backward equations satisfied by the adjoint process are derived.

In addition to the book of Brémaud [5], other works which discuss the control and filtering of jump processes include [3], [4], [6], [12] and [13].

2. Markov Chain.

For $0 \le i \le N$ write $e_i = (0, \dots, 1, \dots, 0)^*$ for the i-th unit (column) vector in R^{N+1}. Consider the Markov process $\{X_t\}$, $t \ge 0$, defined on a probability space (Ω, F, P), whose state space is the set

$$S = \{e_0, e_1, \dots, e_N\}.$$

Write $p_t^i = P(X_t = e_i)$, $0 \le i \le N$. We shall suppose that for some family of matrices A_t, $p_t = (p_t^0, \dots p_t^N)^*$ satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = A_t p_t. \tag{2.1}$$

$A_t = (a_{ij}(t))$, $t \ge 0$, is, therefore, the family of Q matrices of the process. We shall suppose $|a_{ij}(t)| \le B$ for all i, j and $t \ge 0$. Because A_t is a Q-matrix

$$a_{ii}(t) = -\sum_{j \ne i} a_{ji}(t). \tag{2.2}$$

The fundamental transition matrix associated with A will be denoted by $\Phi(t, s)$, so with I the $(N+1) \times (N+1)$ identity matrix

$$\frac{d\Phi(t,s)}{dt} = A_t \Phi(t,s), \qquad \Phi(s,s) = I \tag{2.3}$$

$$\frac{d\Phi(t,s)}{ds} = -\Phi(t,s) A_s, \qquad \Phi(t,t) = I. \tag{2.4}$$

(If A_t is constant $\Phi(t,s) = \exp(t-s)A$.)

Consider the process in state $x \in S$ at time s and write $X_{s,t}(x)$ for its state at the later time $t \ge s$. Then $E[X_{s,t}(x)] = E_{s,x}[X_t] = \Phi(t,s)x$. Write F_t^s for the right continuous, complete filtration generated by $\sigma\{X_r : s \le r \le t\}$, and $F_t = F_t^0$. A basic result (see [8], [11]) is

LEMMA 2.1. *$M_t := X_t - X_0 - \int_0^t A_r X_{r-} dr$ is an $\{F_t\}$ martingale.*

Proof. Suppose $0 \le s \le t$. Then

$$
\begin{aligned}
E[M_t - M_s \mid F_s] &= E\Big[X_t - X_s - \int_s^t A_r X_{r-} dr \mid F_s\Big] \\
&= E\Big[X_t - X_s - \int_s^t A_r X_{r-} dr \mid X_s\Big] \\
&= E\Big[X_t - X_s - \int_s^t A_r X_r dr \mid X_s\Big],
\end{aligned}
$$

because $X_r = X_{r-}$ for each w, except for countably many r,

$$
\begin{aligned}
&= E_{s,X_s}[X_t] - X_s - \int_s^t A_r E_{s,X_s}[X_r] dr \\
&= \Phi(t,s) X_s - X_s - \int_s^t A_r \Phi(r,s) X_s dr = 0 \qquad \text{by (2.3).}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
X_t &= X_0 + \int_0^t A_r X_r dr + M_t \qquad (2.5) \\
&= X_0 + \int_0^t A_r X_{r-} dr + M_t.
\end{aligned}
$$

NOTATION 2.2. If $x = (x_0, x_1, \dots, x_N)^* \in R^{N+1}$ then diag x is the matrix

$$
\begin{pmatrix} x_0 & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_N \end{pmatrix}.
$$

We now give a martingale representation result.

LEMMA 2.3.

$$
X_t = \Phi(t,0)\Big(X_0 + \int_0^t \Phi(r,0)^{-1} dM_r\Big). \tag{2.6}
$$

Proof. The proof follows from (2.5) by variation of constants. Alternatively, differentiate (2.6).

3. Some Basic Martingales.

If x, y are (column) vectors in R^{N+1} we shall write $x \cdot y = x^* y$ for their scalar (inner) product.

Consider $0 \leq i, j \leq N$ with $i \neq j$. Then

$$(X_{s-} \cdot e_i) e_j^* dX_s = (X_{s-} \cdot e_i) e_j^* \Delta X_s$$

$$= (X_{s-} \cdot e_i) e_j^* (X_s - X_{s-}) = I(X_{s-} = e_i,\ X_s = e_j).$$

Define the martingale

$$M_t^{ij} := \int_0^t (X_{s-} \cdot e_i) e_j^* dM_s.$$

(Note the integrand is predictable.) Then

$$M_t^{ij} = \int_0^t (X_{s-} \cdot e_i) e_j^* dX_s - \int_0^t (X_{s-} \cdot e_i) e_j^* A_s X_{s-} ds$$

and, writing $N_t(i,j)$ for the number of jumps of the process X from e_i to e_j up to time t, this is

$$= N_t(i,j) - \int_0^t I(X_{s-} = e_i) a_{ji}(s) ds$$

$$= N_t(i,j) - \int_0^t I(X_s = e_i) a_{ji}(s) ds,$$

because $X_s = X_{s-}$ for each ω, except for countably many s. That is, for $i \neq j$,

$$N_t(i,j) = \int_0^t I(X_s = e_i) a_{ji}(s) ds + M_t^{ij}.$$

For a fixed j, $0 \leq j \leq N$, write $N_t(j)$ for the number of jumps into state e_j up to time t. Then

$$N_t(j) = \sum_{\substack{i=1 \\ i \neq j}}^N N_t(i,j) = \sum_{\substack{i=1 \\ i \neq j}}^N \int_0^t I(X_s = e_i) a_{ji}(s) ds + M_t^j$$

where M_t^j is the martingale $\sum_{\substack{i=1 \\ i \neq j}}^N M_t^{ij}$. Finally, write N_t for the total number of jumps (of any kind) of the process X up to time t. Then

$$N_t = \sum_{j=1}^N N_t(j) = \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq j}}^N \int_0^t I(X_s = e_i) a_{ji}(s) ds + Q_t$$

where Q_t is the martingale $\sum_{j=1}^{N} M_t^j$. However, from (2.2)

$$a_{ii}(s) = -\sum_{\substack{j=1\\ j\neq 1}}^{N} a_{ji}(s)$$

so

$$N_t = -\sum_{i=1}^{N} \int_0^t I(X_s = e_i)a_{ii}(s)ds + Q_t. \tag{3.1}$$

4. Filtering.

A natural problem is the recursive estimation of the state X_t of the Markov chain, given the number of jumps which have occurred to time t. (Formulae for other counting processes such as $N_t(j)$ can be given.) That is, we have on the probability space (Ω, F, P) a

SIGNAL, given by (2.5),

$$X_t = X_0 + \int_0^t A_s X_{s-} ds + M_t \tag{4.1}$$

and an

OBSERVATION PROCESS, given by the counting process

$$N_t = \int_0^t h(s, X_s)ds + Q_t. \tag{4.2}$$

Here

$$h(s, X_s) = -\sum_{i=1}^{N} I(X_s = e_i)a_{ii}(s).$$

NOTATION 4.1. Write $a(s)$ for the vector $(-a_{00}(s),\ -a_{11}(s), \dots,\ -a_{NN}(s))^*$. Then $h(s, X_s) = a(s) \cdot X_s$. We shall further abbreviate $h(s, X_s)$ as $h(s)$.

Recall X_t and N_t are both adapted to the filtration $\{F_t\}$. Write $\{Y_t\}$ for the ri, at continuous, complete filtration generated by N, so $Y_t \subset F_t$ for all t.

NOTATION 4.2. If $\{\phi_t\}$, $t \geq 0$, is any process write $\hat{\phi}$ for the Y-optional projection of ϕ. Then, (see [7], p. 60), for all $t \geq 0$

$$\hat{\phi}_t = E[\phi_t \mid Y_t] \quad \text{a.s.} \tag{4.3}$$

Similarly, write $\hat{\hat{\phi}}$ for the Y-predictable projection of ϕ. Then, for all $t \geq 0$

$$\hat{\hat{\phi}}_t = E[\phi_t \mid Y_{t-}] \quad \text{a.s.} \tag{4.4}$$

REMARKS 4.3. From Theorem 6.48 of [7], for almost all ω

$$\hat{\phi}_t = \hat{\hat{\phi}}_t$$

except for at most countably many values of t. Also, as noted previously, $X_t = X_{t-}$ except for countably many values of t. Therefore,

$$\begin{aligned} \int_0^t \hat{\hat{h}}(r, X_r) du &= \int_0^t \hat{h}(r, X_r) dr \\ &= \int_0^t \hat{h}(r, X_{r-}) dr. \end{aligned}$$

NOTATION 4.4. Write

$$\hat{p}_t = \widehat{X}_t = E[X_t \mid Y_t],$$

and note $\hat{p}_0 = E[X_0] = p_0$, say. Also, as $h(r) = a(r) \cdot X_r$,

$$\hat{h}(r) = a(r) \cdot \hat{p}_r.$$

We shall also introduce the vector

$$h(r)X_r = \text{diag } a(r) \cdot X_r,$$

$$\text{so} \qquad h(r)\hat{X}_r = \text{diag } a(r) \cdot \hat{p}_r.$$

DEFINITION 4.5. The INNOVATION PROCESS is

$$\begin{aligned} \widetilde{Q}_t &:= N_t - \int_0^t \hat{\hat{h}}(r) dr \\ &= N_t - \int_0^t \hat{h}(r) dr = N_t - \int_0^t \hat{h}(r-) dr, \end{aligned}$$

by Remarks 4.3.

It is easily checked that $\widetilde{Q}$ is a Y-martingale. Therefore, we can write

$$N_t = \int_0^t \hat{h}(r-)dr + \widetilde{Q}_t. \tag{4.5}$$

Calculations using Fubini's theorem show that the process

$$\widetilde{M}_t := \hat{p}_t - p_0 - \int_0^t A_s \hat{p}_{s-} ds$$

is a square integrable martingale on the Y-filtration. Therefore, $\widetilde{M}$ can be represented as a stochastic integral with respect to $\widetilde{Q}$,

$$\widetilde{M}_t = \int_0^t \gamma_r d\widetilde{Q}_r$$

where γ is a Y-predictable R^{N+1} valued process such that

$$E\Big[\int_0^\infty |\gamma_r|^2 h(r)dr\Big] < \infty.$$

Therefore, we can write

$$E[X_t \mid Y_t] = \hat{p}_t = p_0 + \int_0^t A_r \hat{p}_{r-} dr + \int_0^t \gamma_r d\widetilde{Q}_r. \tag{4.6}$$

It is known, see [4] or [5], that

$$\gamma_r = I(\hat{p}(r-)\cdot a(r) \neq 0)(\hat{p}(r-)\cdot a(r))^{-1}\{\text{diag } a(r)\cdot \hat{p}(r-) - (\hat{p}(r-)\cdot a(r))\hat{p}(r-) + A_r \hat{p}(r-)\}. \tag{4.7}$$

Therefore $\hat{p}_t = E[X_t \mid Y_t]$ is given by the equation

$$\hat{p}_t = p_0 + \int_0^t A_r \hat{p}_{r-} dr + \int_0^t \gamma_r (dN_r - a(r)\cdot \hat{p}_{r-} dr) \tag{4.8}$$

where γ_r is given by (4.7).

REMARKS 4.6. The disadvantage of (4.8) is that it has the inverse factor $(a(r)\cdot \hat{p}_{r-})^{-1}$. This problem is avoided by considering the related Zakai equation for the unnormalized distribution.

Suppose there is a $k > 0$ such that $-a_{ii}(r) > k$ for all i and $r \geq 0$. Then $h(r)^{-1} = (a(r) \cdot X_r)^{-1} < k^{-1}$ for all $r \geq 0$. Introduce a new measure P_1 on (Ω, F) such that

$$E\Big[\frac{dP_1}{dP} \mid F_t\Big] = \Lambda_t \tag{4.9}$$

where Λ is the martingale

$$\Lambda_t = 1 + \int_0^t \Lambda_{r-}(h(r-)^{-1} - 1)dQ_r. \tag{4.10}$$

It is known, (see [4], [5]) that under P_1 the process N_t is a standard Poisson process. Consider the (P_1, F) martingale

$$\overline{\Lambda}_t = 1 + \int_0^t \overline{\Lambda}_{r-}(h(r-) - 1)d\overline{Q}_r. \tag{4.11}$$

Then it is easily checked that $\Lambda_t \overline{\Lambda}_t = 1$.

To obtain the Zakai equation we take P_1 as the reference probability and compute expectations under P_1. However, it is under measure P that

$$N_t = \int_0^t h(r-)du + Q_t.$$

Write $\Pi(\overline{\Lambda}_t)$ for the Y-optional projection of $\overline{\Lambda}$ under P_1. Then for each $t \geq 0$

$$\Pi(\overline{\Lambda}_t) = E_1[\overline{\Lambda}_t \mid Y_t] \qquad \text{a.s.}$$

Quoting again from [4] or [5] we know that

$$\Pi(\overline{\Lambda}_t) = 1 + \int_0^t \lambda_r d\overline{Q}_r \quad \text{where} \quad \lambda_r = \Pi(\overline{\Lambda}_{r-})(\hat{h}(r-) - 1). \tag{4.12}$$

For any integrable, F_t-measurable, random variable ϕ write

$$\sigma(\phi) = E_1[\overline{\Lambda}_t \phi \mid Y_t].$$

Then

$$\sigma(X_t) = E_1[\overline{\Lambda}_t X_t \mid Y_t] = q_t, \quad \text{say.}$$

Also,

$$\sigma(1) = \Pi(\overline{\Lambda}_t) = 1 + \int_0^t \Pi(\overline{\Lambda}_{r-})(\hat{h}(r-) - 1)d\overline{Q}_r. \tag{4.13}$$

Now q_t is an unnormalized conditional distribution of X_t given Y_t, because

$$\hat{p}_t = E[X_t \mid Y_t] = \sigma(X_t)/\sigma(1) = q_t/\Pi(\overline{\Lambda}_t). \tag{4.14}$$

Note that $\hat{p}_0 = p_0 = q_0$. The Zakai equation for the unnormalized distribution q is, therefore, obtained by calculating the product

$$q_t = \hat{p}_t \cdot \Pi(\overline{\Lambda}_t) \tag{4.15}$$

using (4.8) and (4.13) to obtain:

$$q_t = q_0 + \int_0^t A_r q_{r-} dr + \int_0^t (\text{diag } a(r) - I + A_r) q_{r-} d\overline{Q}_r. \tag{4.16}$$

5. Optimal Control.

Consider a Markov process X, as defined in Section 2, whose state space is the set S of unit vectors $\{e_0, e_1, \ldots, e_N\}$ of R^{N+1}. However, we now suppose that the family of Q-matrix generators $A_t(u)$ depend on a control parameter $u \in U$. Here U, the set of control values, is a compact, convex subset of some Euclidean space R^k. We take $0 \leq t \leq T$ and suppose $A_t(u)$ is measurable on $[0, T] \times U$ and $A_t(\cdot)$ is continuously differentiable on U. Further, we assume $|a_{ij}(t, u)| \leq B$ for $(t, u) \in [0, T] \times U$.

We suppose that only the total number of jumps to time t, N_t, is observed. (The techniques below work for other kinds of observation processes such as $N_t(j, k)$, $N_t(j)$ etc.) $\{F_t\}$, (resp. $\{Y_t\}$), is the right continuous, complete filtration generated by X (resp. N).

CONTROLS 5.1. The set $\underline{U}$ of admissible controls is the set of $\{Y_t\}$-predictable processes with values in U. This means that, if $T_1, T_2, \ldots$ are the jump times of N, then for $T_n < t \leq T_{n+1}$, $u \in \underline{U}$ is a function only of $T_1, T_2, \ldots, T_n$ and t. For each $u \in \underline{U}$, as in Lemma 2.1, M^u is a $(P, \{F_t\})$ martingale where

$$M_t^u := X_t^u - X_0^u - \int_0^t A_r(u) X_{r-}^u dr. \tag{5.1}$$

For $u \in \underline{U}$, write

$$a(s, u) = (-a_{00}(s, u),\ -a_{11}(s, u), \ldots,\ -a_{NN}(s, u))^*$$

and

$$h(s,u) = h(s, X_s^u) = a(s,u) \cdot X_s^u. \tag{5.2}$$

Then with $\widehat{}$ again denoting the Y-optional projection under P

$$\hat{p}_s(u) = \widehat{X}_s^u = E[X_s^u \mid Y_s]$$

and $\hat{h}(s,u) = a(s,u) \cdot \hat{p}_s(u)$. Also,

$$h(s,u) \cdot X_s^u = \text{diag } a(s,u) \cdot X_s^u$$

and

$$\widehat{h(s,u) \cdot X_s^u} = \text{diag } a(s,u) \cdot \hat{p}_s(u).$$

N_t can, therefore, be written

$$N_t = \int_0^t h(s,u) ds + Q_t^u \tag{5.3}$$

$$= \int_0^t \hat{h}(s-,u) ds + \widetilde{Q}_t^u. \tag{5.4}$$

Unlike the situation considered in [9], the noises in the signal and observation processes are correlated, $[X^u, N]_t \neq 0$.

COST 5.2. A real function on the state space $S = \{e_0, e_1, \ldots, e_N\}$ is represented by a vector $\ell = (\ell_0, \ldots, \ell_N)^* \in R^{N+1}$. Write $\langle \ell, x \rangle = \ell^* \cdot x$ for the inner product on R^{N+1}. The control problem we wish to consider is that of choosing $u \in \underline{U}$ so that the expected cost

$$J(u) = E[\langle \ell, X_T^u \rangle]$$

is minimized.

NOTATION 5.3. Write P_1, as in Section 4, for the probability measure under which N is a standard Poisson process.

For each $u \in \underline{U}$ introduce the (P_1, F) martingale

$$\overline{\Lambda}_t^u = 1 + \int_0^t \overline{\Lambda}_{r-}^u (h(r-,u) - 1) d\overline{Q}_r$$

and write $\Pi(\overline{\Lambda}_t^u)$ for the Y-optional projection of $\overline{\Lambda}$ under P_1. Then, $E_1\left[\frac{dP}{dP_1} \mid F_t\right] = \overline{\Lambda}_t^u$ and with

$$\sigma(X_t^u) = E_1[\overline{\Lambda}_t^u X_t^u \mid Y_t] = q_t(u)$$

we have, as in (4.15),

$$q_t(u) = \Pi(\overline{\Lambda}_t^u)\hat{p}_t(u).$$

Furthermore, with $A_t(u) = (a_{ij}(t,u))$ write

$$B_t(u) = (\text{diag }(a(t,u)) - I + A_t(u))$$

$$= \begin{pmatrix} -1 & a_{01}(t,u) & \dots & a_{0N}(t,u) \\ a_{10}(t,u) & -1 & \dots & a_{1N}(t,u) \\ \vdots & & & \\ a_{N0}(t,u) & \dots & \dots & -1 \end{pmatrix}.$$

Then, for $u \in \underline{U}$, the unnormalized distribution $q_t(u)$ is given by the Zakai equation

$$q_t(u) = p_0 + \int_0^t A_r(u)q_{r-}(u)du + \int_0^t B_r(u)q_{r-}(u)d\overline{Q}_r.$$

Here $\overline{Q}$ is a standard Poisson process under P_1.

The expected cost if $u \in \underline{U}$ is used is

$$J(u) = E[\langle \ell, X_T^u \rangle] = E_1[\overline{\Lambda}_T^u \langle \ell, X_T^u \rangle] = E_1[\langle \ell, \overline{\Lambda}_T^u X_T^u \rangle]$$

$$= E_1[\langle \ell, E_1[\overline{\Lambda}_T^u X_T^u \mid Y_T] \rangle] = E_1[\langle \ell, q_T^u \rangle].$$

The control problem has, therefore, been formulated in separated form: find $u \in \underline{U}$ which minimizes

$$J(u) = E_1[\langle \ell, q_T^u \rangle]$$

where for $0 \le t \le T$, $q_t(u)$ satisfies the dynamics

$$q_t(u) = p_0 + \int_0^t A_r(u)q_{r-}(u)du + \int_0^t B_r(u)q_{r-}(u)d\overline{Q}_r. \tag{5.5}$$

Under the measure P_1, N is a standard Poisson process, so $\overline{Q}_t = N_t - t$ is a (P_1, Y) martingale. Furthermore, the controls $u \in \underline{U}$ are in the 'stochastic open loop' form discussed by Bismut [2] and Kushner [10]. That is, the controls are adapted to the filtration, as described above, and are not explicitly functions of the state q.

6. Differentiation.

NOTATION 6.1. For $u \in \underline{U}$ write $\Phi^u(t,s)$ for the fundamental matrix solution of

$$d\Phi^u(t,s) = A_t(u)\Phi^u(t-,s)dt + B_t(u)\Phi^u(t-,s)(dN_t - dt) \tag{6.1}$$

with initial condition $\Phi^u(s,s) = I$, the $(N+1)\times(N+1)$ identity matrix.

Note that $A_t(u) - B_t(u) = \text{diag}\,(1 + a_{ii}(t,u))$ and write

$$D^u(s,t) = \text{diag}\left(\exp \int_s^t (1 + a_{ii}(r,u))dr\right).$$

Then if $T_n \le t < T_{n+1}$,

$$\Phi^u(t,0) = D^u(t,T_n)\big(I + B_{T_n}(u_{T_n})\big)D^u(T_n,T_{n-1})\big(I + B_{T_{n-1}}(u_{T_{n-1}})\big)$$
$$\ldots\big(I + B_{T_1}(u_{T_1})\big)D^u(T_1,0). \tag{6.2}$$

The matrices $D^u(s,t)$ have inverses

$$\text{diag}\left(\exp - \int_s^t (1 + a_{ii}(r,u))dr\right);$$

we make the following assumption:

ASSUMPTION 6.2. For $u \in \underline{U}$ and $t \in [0,T]$ the matrix $(I + B_t(u_t))$ is nonsingular.

The matrix Φ is the analog of the Jacobian in the continuous case. We now derive the equation satisfied by the inverse Ψ of Φ.

LEMMA 6.3. *For $u \in \underline{U}$ consider the matrix Ψ^u defined by the equation*

$$\Psi^u(t,s) = I - \int_s^t \Psi^u(r-,s)A_r(u)dr - \int_s^t \Psi^u(r-,s)B_r(u)d\overline{Q}_r$$
$$+ \int_s^t \Psi^u(r-,s)B_r^2(u)(I + B_r(u))^{-1}dN_r. \tag{6.3}$$

Then $\Psi^u(t,s)\Phi^u(t,s) = I$ for $t \ge s$.

Proof. Recall

$$\Phi^u(t,s) = I + \int_s^t A_r(u)\Phi^u(r-,s)dr + \int_s^t B_r(u)\Phi^u(r-,s)d\overline{Q}_r. \tag{6.4}$$

Then by the product rule

$$\begin{aligned}\Psi\Phi = I &+ \int_s^t \Psi A\Phi dr + \int_s^t \Psi B\Phi d\overline{Q}_r \\ &- \int_s^t \Psi A\Phi dr - \int_s^t \Psi B\Phi d\overline{Q}_r + \int_s^t \Psi B^2(I+B)^{-1}\Phi dN_r \\ &- \int_s^t \Psi B^2\Phi dN_r + \int_s^t \Psi B^2(I+B)^{-1}B\Phi dN_r \\ = I&,\end{aligned}$$

as the integral terms cancel. □

We shall suppose there is an optimal control $u^* \in \underline{U}$. Write q^* for q^{u^*}, Φ^* for Φ^{u^*} etc. Consider any other control $v \in \underline{U}$. Then for $\theta \in [0,1]$,

$$u_\theta(t) = u^*(t) + \theta(v(t) - u(t)) \in \underline{U}.$$

Because $U \subset R^k$ is compact, the set $\underline{U}$ of admissible controls can be considered as a subset of the Hilbert space $H = L^2[\Omega \times [0,T] : R^k]$. Now

$$J(u \quad \geq J(u^*). \tag{6.5}$$

Therefore, if the Gâteaux derivative $J'(u^*)$ of J, as a functional on the Hilbert space H, is well defined, differentiating (6.5) in θ, and evaluating at $\theta = 0$, implies

$$\langle J'(u^*), v(t) - u^*(t)\rangle \geq 0$$

for all $v \in \underline{U}$.

LEMMA 6.4. *Suppose $v \in \underline{U}$ is such that $u^*_\theta = u^* + \theta v \in \underline{U}$ for $\theta \in [0,\alpha]$. Write $q_t(\theta)$ for the solution $q_t(u^*_\theta)$ of (5.5). Then $z_t = \frac{\partial q_t(\theta)}{\partial\theta}\Big|_{\theta=0}$ exists and is the unique solution of the equation*

$$\begin{aligned} z_t = &\int_0^t \Big(\frac{\partial A}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dr + \int_0^t A_r(u^*)z_{r-}dr \\ &+ \int_0^t \Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}d\overline{Q}_r + \int_0^t B_r(u^*)z_{r-}d\overline{Q}_r. \end{aligned}\tag{6.6}$$

Proof. $q_t(\theta) = p_0 + \int_0^t A_r(u^* + \theta v)q_{r-}(\theta)dr + \int_0^t B_r(u^* + \theta v)q_{r-}(\theta)d\overline{Q}_r$. The stochastic integrals are defined pathwise, so differentiating under the integrals gives the result. Comparing (6.4) and (6.6) we have the following result by variation of constants.

LEMMA 6.5. *Write*

$$\eta_{0,t} = \int_0^t \Psi^*(r-,0)\Big(\frac{\partial A}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dr$$
$$+ \int_0^t \Psi^*(r-,0)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}d\overline{Q}_r$$
$$- \int_0^t \Psi^*(r-,0)(I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dN_r. \tag{6.7}$$

Then $z_t = \Phi^*(t,0)\eta_{0,t}$.

Proof. Using the differentiation rule

$$\Phi^*(t,0)\eta_{0,t} = \int_0^t \Phi^*_- \cdot d\eta + \int_0^t d\Phi^*\eta_- + [\Phi,\eta]_t.$$

Because $\Phi^*_-\Psi^*_- = I$, therefore

$$\Phi^*(t,0)\eta_{0,t} = \int_0^t \Big(\frac{\partial A}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dr$$
$$+ \int_0^t \Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}d\overline{Q}_r$$
$$- \int_0^t (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dN_r$$
$$+ \int_0^t A_r(u)\Phi^*(r-,0)\eta_{0,r-}dr + \int_0^t B_r(u)\Phi^*(r-,0)\eta_{0,r-}d\overline{Q}_r$$
$$+ \int_0^t B_r(u)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dN_r$$
$$- \int_0^t B_r(u)(I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q^*_{r-}dN_r.$$

Now the dN integrals sum to 0, showing that $\Phi^*\eta$ satisfies the same equation (6.7) as z. Consequently, by uniqueness, the result follows.

COROLLARY 6.6. $\frac{dJ}{d\theta}(u^*_\theta)\Big|_{\theta=0} = E_1[\langle \ell, \Phi^*(T,0)\eta_{0,T}\rangle]$.

Proof. $J(u^*_\theta) = E_1[\langle \ell, q_T(\theta) >]$. The result follows from Lemmas 6.4 and 6.5.

NOTATION 6.7. Write $\Phi^*(T,0)'$ for the transpose of $\Phi^*(T,0)$ and consider the square integrable, vector martingale

$$M_t := E_1[\Phi^*(T,0)'\ell \mid Y_t].$$

Then M_t has a representation as a stochastic integral

$$M_t = E_1[\Phi^*(T,0)'\ell] + \int_0^t \gamma_r d\overline{Q}_r$$

where γ is a predictable R^{N+1} valued process such that

$$\int_0^T E_1[\gamma_r^2]dr < \infty.$$

Under a Markov hypothesis γ will be explicitly determined below.

DEFINITION 6.8. The adjoint process is

$$p_t := \Psi^*(t,0)'M_t.$$

THEOREM 6.9.

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = \int_0^T E_1\Big[\Big\langle p_{r-}, \Big\{\Big(\frac{\partial A}{\partial u}(r,u^*) - (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)\Big\}v_r q_{r-}^*\Big\rangle$$
$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle\Big]dr. \tag{6.8}$$

Proof. First note that

$$\begin{aligned}\langle M_T, \eta_{0,T}\rangle = &\int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)\Big(\frac{\partial A}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle dr \\ &+ \int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle d\overline{Q}_r \\ &- \int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)(I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle dN_r \\ &+ \int_0^T \Big\langle \gamma_r, \eta_{0,r-}\Big\rangle d\overline{Q}_r + \int_0^T \Big\langle \gamma_r \Psi^*(r-,0)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle dN_r \\ &- \int_0^T \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle dN_r. \end{aligned} \tag{6.9}$$

Taking expectations under P, we have

$$\left.\frac{dJ(u_\theta^*)}{d\theta}\right|_{\theta=0} = E_1[\langle \ell, \Phi^*(T,0)\eta_{0,T}\rangle]$$

$$= E_1[\langle \Phi^*(T,0)'\ell, \eta_{0,T}\rangle]$$

$$= E_1[\langle M_T, \eta_{0,T}\rangle].$$

Combining the last two terms in (6.9) and using the fact that $N_t - t$ is a P_1 martingale, this is

$$= \int_0^T E_1\Big[\Big\langle p_{r-}, \Big(\frac{\partial A}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle$$

$$- \Big\langle p_{r-}, (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle$$

$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)v_r q_{r-}^*\Big\rangle\Big]dr.$$

□

Now consider perturbations of u^* of the form

$$u_\theta(t) = u^*(t) + \theta(v(t) - u(t))$$

for $\theta \in [0,1]$ and any $v \in \underline{U}$. Then as noted above

$$\left.\frac{dJ(u_\theta)}{d\theta}\right|_{\theta=0} = \langle J'(u^*), v(t) - u^*(t)\rangle \geq 0.$$

Expression (6.8) is, therefore, true when v is replaced by $v - u^*$ for any $v \in \underline{U}$, and we can deduce the following minimum principle.

THEOREM 6.10. *Suppose $u^* \in \underline{U}$ is an optimal control. Then* a.s. *in w and* a.e. *in t*

$$\Big\langle p_{r-}, \Big\{\Big(\frac{\partial A}{\partial u}(r,u^*) - (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)\Big\}(v_r - u_r^*)q_{r-}^*\Big\rangle$$

$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)(v_r - u_r^*)q_{r-}^*\Big\rangle \geq 0. \quad (6.10)$$

7. The Equation for the Adjoint Process.

The process p is the adjoint process. However, (6.10) also contains the integrand γ. In this section we shall obtain a more explicit expression for γ in the case when u^* is Markov, and also derive forward and backward equations satisfied by p.

ASSUMPTION 7.1 The optimal control u^* is a Markov, feedback control. That is, $u^* : [0, T] \times R^{N+1} \to U$ so that $u^*(s, q^*_{s-}) \in U$.

Note that if u_m is a Markov control, with a corresponding solution $q_t(u_m)$ of (5.5), then u_m can be considered as a stochastic open loop control $u_m(\omega)$ by setting

$$u_m(\omega) = u_m(s, q^*_{s-}(u_m)(\omega)).$$

This means the control u_m "follows" the 'left limit' of its original trajectory $q_s(u_m)$ rather than any new trajectory.

LEMMA 7.2. *Write δ for the predictable "integrand" such that*

$$\Delta p_t = p_t - p_{t-} = \delta_t \Delta N_t,$$

$$\text{i.e.,} \qquad p_t = p_{t-} + \delta_t \Delta N_t.$$

Furthermore, write

$$q_{t-} = q,$$

$$B_{t-}(u^*(t-, q)) = B^*(q_{t-}) = B^*(q),$$

and

$$B_t(u^*(t, q_t)) = B^*(q_t).$$

Then

$$\delta_t(q) = (I + B^*((I + B^*(q))q))^{-1} p_{t-}((I + B^*(q))q) - p_{t-}(q). \tag{7.1}$$

Proof. Let us examine what happens if there is a jump at time t; that is, suppose $\Delta N_t = 1$. Then from (5.5)

$$q_t = (I + B^*(q))q.$$

By the Markov property and from (6.2) and Definition 6.8,

$$\begin{aligned}
p_t &= E[D^*(t,T_k)(I+B'_{T_k}(u^*))\dots D^*(T,T_N)\ell \mid Y_t] \\
&= p_t(q_t) \\
&= p_t((I+B^*(q))q) \\
&= (I+B^*(q_t)')^{-1}p_{t-}((I+B^*(q))q),
\end{aligned}$$

and the result follows. Heuristically, the integrand δ assumes there is a jump at t; the question of whether there is a jump is determined by the factor ΔN_t.

THEOREM 7.3. *Under Assumption 7.1 and with δ_t given by (7.1)*

$$\gamma_r = \Phi^*(r-,0)'((I+B'_r(u^*))\delta_r + B'_r(u^*)p_{r-}). \tag{7.2}$$

Proof. $\Phi^*(t,0)'p_t = M_t = E_1[\Phi^*(T,0)'\ell \mid Y_t] = E_1[\Phi^*(T,0)'\ell] + \int_0^t \gamma_r d\overline{Q}_r$. However, if u^* is Markov the process q^* is Markov, and, writing $q = q_t^*$, $\Phi = \Phi^*(t,0)$,

$$\begin{aligned}
E_1[\Phi^*(T,0)'\ell \mid Y_t] &= E_1[\Phi'\Phi^*(T,t)'\ell \mid q,\Phi] \\
&= \Phi' E_1[\Phi^*(T,t)'\ell \mid q].
\end{aligned}$$

Consequently, $p_t = E_1[\Phi^*(T,t)'\ell \mid q]$ is a function only of q, so by the differentiation rule:

$$\begin{aligned}
p_t &= p_0 + \int_0^t \frac{\partial p_{r-}}{\partial q}(Aq_{r-}dr + Bq_{r-}d\overline{Q}_r) + \int_0^t \frac{\partial p_{r-}}{\partial r}\, dr \\
&\quad + \sum_{0<r\le t}\left(p_r - p_{r-} - \frac{\partial p_{r-}}{\partial q}Bq_{r-}\Delta N_r\right) \\
&= p_0 + \int_0^t \left[\frac{\partial p_{r-}}{\partial q}(Aq_{r-} - Bq_{r-}) + \delta_r\right]dr + \int_0^t \delta_r d\overline{Q}_r.
\end{aligned}$$

Evaluating the product:

$$\begin{aligned}
M_t = \Phi^*(t,0)'p_t &= p_0 + \int_0^t \Phi^*(r-,0)'\left[\frac{\partial p_{r-}}{\partial q}(Aq_{r-} - Bq_{r-}) + \delta_r\right]dr \\
&\quad + \int_0^t \Phi^*(r-,0)'\frac{\partial p_{r-}}{\partial r}\, dr + \int_0^t \Phi^*(r-,0)'\delta_r d\overline{Q}_r + \int_0^t \Phi^*(r-,0)'A'p_{r-}dr \\
&\quad + \int_0^t \Phi^*(r-,0)'B'p_{r-}d\overline{Q}_r + \int_0^t \Phi^*(r-,0)'B'\delta_r d\overline{Q}_r + \int_0^t \Phi^*(r-,0)'B'\delta_r dr.
\end{aligned} \tag{7.3}$$

However, M_t is a martingale, so the sum of the dr integrals in (7.3) must be 0, and

$$\gamma_r = \Phi^*(r-,0)'(\delta_r + B_r'(u_r^*)\delta_r + B_r'(u_r^*)p_{r-}).$$

□

THEOREM 7.4. *Suppose the optimal control u^* is Markov. Then* a.s. *in ω and* a.e. *in t, u^* satisfies the minimum principle*

$$\left\langle p_{r-}, \frac{\partial A}{\partial u}(r,u^*)(v_r - u_r^*)q_{r-}^* \right\rangle + \left\langle \delta_r, \frac{\partial B}{\partial u}(r,u^*)(v_r - u_r^*)q_{r-}^* \right\rangle \geq 0. \tag{7.4}$$

Proof. Substituting γ from (7.2) into (6.10), and noting $B(I+B)^{-1}-(I+B)^{-1}B = 0$, the result follows. (Substituting for B and δ gives an alternative form.)

We now derive a forward equation satisfied by the adjoint process p:

THEOREM 7.5. *With δ given by (7.1)*

$$p_t = E_1[\Phi^*(T,0)'\ell] - \int_0^t A_r'(u_r^*)p_{r-}dr - \int_0^t (I + B_r'(u_r^*))\delta_r dr + \int_0^t \delta_r dN_r. \tag{7.5}$$

Proof. $p_t = \Psi^*(t,0)'M_t$ and from (6.3) this is

$$= E_1[\Phi^*(T,0)'\ell] - \int_0^t A'\Psi^{*\prime}Mdr - \int_0^t B'\Psi^{*\prime}Md\overline{Q}_r + \int_0^t (I+B')^{-1}B'^2\Psi^{*\prime}MdN_r$$

$$+ \int_0^t \Psi^{*\prime}\gamma_r d\overline{Q}_r - \int_0^t B'\Psi^{*\prime}\gamma_r dN_r + \int_0^t (I+B')^{-1}B'^2\Psi^{*\prime}\gamma_r dN_r$$

$$= E_1[\Phi^*(T,0)'\ell] - \int_0^t A'p_{r-}dr - \int_0^t B'p_{r-}d\overline{Q}_r + \int_0^t (I+B')^{-1}B'^2 p_{r-}dN_r$$

$$+ \int_0^t ((I+B')\delta_r + B'p_{r-})d\overline{Q}_r - \int_0^t (I+B')^{-1}B'((I+B')\delta_r + B'p_{r-})dN_r$$

$$= E_1[\Phi^*(T,0)'\ell] - \int_0^t A'p_{r-}dr + \int_0^t (I+B')\delta_r d\overline{Q}_r - \int_0^t B'\delta_r dN_r$$

and the result follows. □

However, an alternative backward equation for the adjoint process p is obtained from the observation that the sum of the bounded variation terms in (7.3) must be identically zero. Therefore, we have the following result which appears to be new:

THEOREM 7.6. *With δ given by (7.1) the Markov adjoint process $p_t(q)$ is given by the backward equation*

$$\frac{\partial p_t}{\partial t} + \frac{\partial p_t}{\partial q} \cdot (A^*(q)' - B^*(q)')q + A^*(q)'p_t + (I + B^*(q)')\delta_t = 0$$

with the terminal condition

$$p_T = \ell.$$

8. Conclusion.

A finite state space Markov chain was considered. Without loss of generality its state space was taken to be the set of unit basis vectors of R^{N+1}. Basic martingales associated with the Markov chain were identified and the solution to filtering problems given when only the total number of jumps are observed. On the basis of knowing only the total number of jumps a control problem associated with the Markov chain is discussed in 'separated' form. That is the Zakai equation for the unnormalized distributions is obtained. This is a linear, vector equation driven by a standard Poisson process in which (unlike earlier work on controlled Markov chains) the control variable also appears in the 'diffusion' coefficient multiplying the noise term. By adapting techniques of Bensoussan and calculating a Gâteaux derivative the minimum principle satisfied by an optimal control is obtained. Finally, in the case when the optimal control is Markov, the integrand in a martingale representation can be obtained explicitly, and forward and backward equations satisfied by the adjoint process derived.

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INTEGRATION BY PARTS
FOR THE SINGLE JUMP PROCESS

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ABSTRACT: By considering small perturbations in time, which are then compensated by changing the measure, a new integration-by-parts formula is obtained for functionals of a single jump process. For martingales associated with observing the time of the jump a new expression is derived for the integrand when they are represented as stochastic integrals.

KEY WORDS: Single jump process, martingale, stochastic integral, integration by parts.

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Integration by Parts for the Single Jump Process

Robert J. Elliott and Allanus H. Tsoi

1. Introduction.

Integration by parts has played a basic role in the Malliavin calculus and its applications, particularly in the work of Bismut (1981). In this paper the concept is investigated in the fundamental situation of a stochastic process with a single random jump. When the state space of the process is Euclidean space (or, possibly, an open, non-empty subset of a Euclidean space) the techniques of Norris (1988) can be specialized to the single jump situation. This method, described in Section 2, considers a small ε-perturbation in the state space of the process. The effect of the perturbation can be removed by a Girsanov change of measure, and the integration by parts formula is obtained by differentiating in ε.

However, for a process whose state space is a general measure space, the perturbation of the kind considered by Norris may not make sense. Such processes include those with discrete state spaces, and, in particular, the process which observes a single random instant at a time T. In the latter case the process $p_t = I_{t \geq T}$ takes only the values 0 or 1.

For general jump processes, therefore, an alternative ε-perturbation in the time direction is introduced. By differentiating a new integration by parts formula, which involves a time derivative, is obtained. In the case of the fundamental process p_t an alternative expression for the integrand in a martingale representation result is derived.

2. Integration by Parts for $\mathbb{R}^d$-Valued Single Jump Processes.

Consider a single jump process with state space $\mathbb{R}^d$ for some $d \geq 1$, which remains at its initial position z_0 until a random time T, when it jumps to a new random position Z.

The underlying probability space is taken as $([0,\infty]\times \mathbb{R}^d,\ \mathcal{B}([0,\infty])\times\mathcal{B}(\mathbb{R}^d),\mu)$. For $t\geq 0$, let $\mathcal{F}_t$ be the completed σ-field generated by the process up to time t. Suppose (λ,Λ) is the Lévy system for the process (see Elliott (1982)). For $A\in\mathcal{B}(\mathbb{R}^d)$, let

$$p(t,A)=I_{t\geq T}I_{Z\in A} \tag{2.1}$$

$$\tilde{p}(t,A)=-\int_{]0,t\wedge T]}\lambda(s,A)\frac{dF_s}{F_{s-}} \tag{2.2}$$

where $F_t=\mu(]t,\infty]\times\mathbb{R}^d)$. Then $q(t,A)=p(t,A)-\tilde{p}(t,A)$ is an $\mathcal{F}_t$-martingale.

We assume that F_t and λ are absolutely continuous, so that there exist functions f_s and $g(y)>0$ such that

$$dF_s=f_s ds$$

$$\lambda(s,dy)=g(y)dy.$$

Consequently,

$$\tilde{p}(ds,dy)=\begin{cases}-g(y)\ \frac{f_s}{F_s}\ dyds & \text{if } s\leq T\\ 0 & \text{if } s>T.\end{cases} \tag{2.3}$$

Let $v(t,y)$ be an $\mathbb{R}^d$-valued function which satisfies:

(i) $v(t,\cdot)$ is C^1 for each $t\geq 0$; v and $\frac{\partial}{\partial y}\ v(t,y)$ are uniformly bounded.

(ii) supp $v(\cdot,\cdot)\subseteq[0,\infty)\times K$ for some compact $K\subseteq\mathbb{R}^d$.

For small $\varepsilon\in\mathbb{R}$ and any $\phi\in L^1(\mu)$, define p^ε by:

$$\int_0^t\int_E\phi(s,y)p^\varepsilon(ds,dy)=\int_0^t\int_E\phi(s,\theta^\varepsilon(s,y))p(ds,dy), \tag{2.4}$$

where

$$\theta^\varepsilon(t,y)=y+\varepsilon v(t,y).$$

Set

$$\lambda^\varepsilon(t,y) = \frac{\partial\theta^\varepsilon(t,y)}{\partial y}\ \frac{g(\theta^\varepsilon(t,y))}{g(y)} \tag{2.5}$$

and

$$X_t = \int_0^t \int_E (\lambda^\varepsilon(s,y) - 1) q(ds,dy). \tag{2.6}$$

Define the family $\{Z_t^\varepsilon,\ t \geq 0\}$ of exponentials by:

$$\begin{aligned} Z_t^\varepsilon &= \exp(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t) \prod_{0\leq s\leq t} (1 + \Delta X_s) e^{-\Delta X_s} \\ &= \exp\Big(\int_0^t \int_E \log \lambda^\varepsilon(s,y) dp - \int_0^t \int_E (\lambda^\varepsilon(s,y) - 1) d\tilde{p}\Big). \end{aligned} \tag{2.7}$$

Then Z_t^ε satisfies:

$$Z_t^\varepsilon = \int_0^t \int_E Z_{s-}^\varepsilon (\lambda^\varepsilon(s,y) - 1) q(ds,dy) \tag{2.8}$$

and $\{Z_t^\varepsilon,\ t \geq 0\}$ is a martingale with $E[Z_t^\varepsilon] = 1$.

Define a new probability measure μ^ε by:

$$\frac{d\mu^\varepsilon}{d\mu} = Z_t^\varepsilon \quad \text{on} \quad \mathcal{F}_t.$$

LEMMA 2.1. *Under μ^ε, p^ε has the original law of p.*

Proof. It suffices to check for test functions $\phi \in L^1(\mu)$ and for

$$\begin{aligned} U_t^\varepsilon &= \exp\Big\{\int_0^t \int_E \phi(s,y) p^\varepsilon(ds,dy)\Big\} Z_t^\varepsilon \\ &= \exp\Big\{\int_0^t \int_E \phi(s, \theta^\varepsilon(s,y)) p(ds,dy)\Big\} Z_t^\varepsilon \end{aligned}$$

that $E[U_t^\varepsilon]$ does not depend on ε. Let

$$Y_t = \exp\Big\{\int_0^t \int_E \phi(s, \theta^\varepsilon(s,y)) p(ds,dy)\Big\}.$$

By the differentiation rule,

$$U_t^\varepsilon = 1 + \int_0^t \int_E Y_{s-} dZ_s^\varepsilon + \int_0^t \int_E Z_{s-}^\varepsilon dY_s + [Y, Z^\varepsilon]_t.$$

But

$$\int_0^t \int_E Z_{s-}^\varepsilon dY_s = \int_0^t \int_E U_{s-}^\varepsilon [\exp(\phi(s, \theta^\varepsilon(s,y))) - 1] p(ds, dy)$$

$$\Delta Y_s = Y_{T-} [\exp\{\phi(T, \theta^\varepsilon(T, Z))\} - 1] I_{s=T}$$

$$\Delta Z_s^\varepsilon = Z_{T-}^\varepsilon [\lambda^\varepsilon(T, Z) - 1] I_{s=T}.$$

Hence,

$$[Y, Z^\varepsilon]_t = \Delta Y_T \Delta Z_T^\varepsilon I_{t \geq T}$$

$$= U_{T-}^\varepsilon [\exp\{\phi(T, \theta^\varepsilon(T, Z))\} - 1][\lambda^\varepsilon(T, Z) - 1] I_{t \geq T}$$

$$= \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1][\lambda^\varepsilon(s,y) - 1] p(ds, dy).$$

Hence,

$$U_t^\varepsilon = 1 + \text{Martingale} + \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1] \lambda^\varepsilon(s,y) p(ds, dy)$$

$$= 1 + \text{Martingale} + \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1] \lambda^\varepsilon(s,y) \tilde{p}(ds, dy)$$

$$= 1 + \text{Martingale} - \int_0^t \int_E U_{s-}^\varepsilon [\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1] \lambda^\varepsilon(s,y) g(y) \frac{f_s}{F_s} \, dyds.$$

Thus

$$E[U_t^\varepsilon] = 1 - \int_0^t \int_E E[U_s^\varepsilon][\exp\{\phi(s, \theta^\varepsilon(s,y))\} - 1] g(\theta^\varepsilon(s,y)) \frac{\partial \theta^\varepsilon(s,y)}{\partial y} \frac{f_s}{F_s} \, dyds$$

$$= 1 - \int_0^t \int_E E[U_s^\varepsilon][\exp\{\phi(s,y)\} - 1] g(y) \frac{f_s}{F_s} \, dyds$$

by the Jacobian formula. Thus $E[U_t^\varepsilon]$ is independent of ε. $\square$

As a consequence of Lemma 2.1, we have

$$E[Z_T^\varepsilon \exp\{\phi(T, Z + \varepsilon V(T, Z))\}] = E[\exp\{\phi(T, Z)\}] \tag{2.9}$$

which leads us to the following theorem:

THEOREM 2.2. *Suppose $G : [0, \infty] \times \mathbb{R}^d \to \mathbb{R}$ is positive, bounded and that its partial derivative $\dfrac{\partial G(t, z)}{\partial z}$ exists and is bounded. Then*

$$E\Big[\Big(\int_0^T \int_E \Big(\frac{\partial}{\partial y} V(t, y) + \frac{g'(y)}{g(y)} V(t, y)\Big) q(ds, dy)\Big) G(T, Z)\Big] = -E\Big[\frac{\partial G(T, Z)}{\partial z} V(T, Z)\Big]. \tag{2.10}$$

Proof. Differentiate (2.9) with respect to ε, then set $\varepsilon = 0$ to obtain

$$\begin{aligned} &E\Big[\frac{d}{d\varepsilon} Z_T^\varepsilon\Big|_{\varepsilon=0} \exp\{\phi(T, Z + \varepsilon V(T, Z))\}\Big|_{\varepsilon=0}\Big] \\ &+E\Big[Z_T^\varepsilon\Big|_{\varepsilon=0} \frac{d}{d\varepsilon} \exp\{\phi(T, Z + \varepsilon V(T, Z))\}\Big|_{\varepsilon=0}\Big] = 0. \end{aligned} \tag{2.11}$$

From (2.8),

$$\begin{aligned} \frac{d}{d\varepsilon} Z_T^\varepsilon = &\int_0^T \int_E \frac{dZ_{t-}^\varepsilon}{d\varepsilon} (\lambda^\varepsilon(t, y) - 1) q(dt, dy) \\ &+ \int_0^T \int_E Z_{t-}^\varepsilon \frac{d}{d\varepsilon} \lambda^\varepsilon(t, y) q(dt, dy). \end{aligned}$$

From the definition of $\lambda^\varepsilon(t, y)$,

$$\lambda^\varepsilon(t, y)\Big|_{\varepsilon=0} = 1$$

and from (2.7),

$$Z_{t-}^\varepsilon\Big|_{\varepsilon=0} = 1.$$

Also,

$$\frac{d\lambda^{\varepsilon}(t,y)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y).$$

Hence

$$\frac{d}{d\varepsilon}\, Z_T^{\varepsilon}\bigg|_{\varepsilon=0} = \int_0^T \int_E \Big(\frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y)\Big) q(dt,dy).$$

Thus (2.11) becomes

$$\begin{aligned} E\Big[\Big(\int_0^T \int_E \Big(\frac{\partial}{\partial y}\, v(t,y) + \frac{g'(y)}{g(y)}\, v(t,y)\Big) q(dt,dy)\Big) \exp(\phi(T,Z))\Big] \\ = -E\Big[\exp\{\phi(T,Z)\}\Big(\frac{\partial}{\partial z}\phi(T,Z)\Big) V(T,Z)\Big]. \end{aligned} \tag{2.12}$$

Let $\phi(T,Z) = \log G(T,Z)$. Then (2.12) becomes (2.10) and the proof is complete. □

3. Integration by Parts for a General Jump Process.

Consider a single jump process with values in a Lusin space $(E,\mathcal{E})$. The underlying probability space is $([0,\infty] \times E,\ \mathcal{B}([0,\infty]) \times \mathcal{E},\ \mu)$. In this section we suppose that for every $t \geq 0$, $F_t > 0$, and both F_t and Λ_t are continuous in t. Furthermore, we assume that there exists a function $\alpha(s)$, with $\alpha(s) > 0$ for all $s \geq 0$, such that

$$\Lambda_t = \int_0^t \alpha(s)ds.$$

Let $u : [0,\infty] \times E \to \mathbb{R}$ be a bounded, positive, deterministic function such that

$$u_s(y) = 0 \quad \text{if} \quad s \notin [0,b]$$

for some fixed $b \in \mathbb{R}$. For $\varepsilon > 0$, define

$$\Lambda_t^{\varepsilon} = \int_0^t \int_E (1 + \varepsilon u_s(y))\lambda(s,dy)d\Lambda_s. \tag{3.1}$$

Consider the new measure μ^ε which has a Lévy system $(\lambda, \Lambda^\varepsilon)$. Then (see Elliott (1982)) $\mu^\varepsilon \ll \mu$, and if

$$L^\varepsilon = \frac{d\mu^\varepsilon}{d\mu},$$

we have

$$L^\varepsilon(t) = \int_E (1 + \varepsilon u_t(y))\lambda(t, dy) \exp\Big\{ - \int_0^t \int_E \varepsilon u_s(y))\lambda(s, dy) d\Lambda_s \Big\}. \tag{3.2}$$

Furthermore, if $L_t^\varepsilon = E[L^\varepsilon(t) \mid \mathcal{F}_t]$ then $\{L_t^\varepsilon,\ t \geq 0\}$ satisfies

$$\begin{aligned} L_t^\varepsilon &= 1 + \int_0^t L_{s-}^\varepsilon dM_s \\ &= 1 + \int_0^t L_{s-}^\varepsilon \int_E \varepsilon u_s(y)\lambda(s, dy) q(ds, E), \end{aligned} \tag{3.3}$$

where

$$M_t = \int_0^t \int_E \varepsilon u_s(y)\lambda(s, dy) q(ds, E).$$

If $F_t^\varepsilon = \mu^\varepsilon(]t, \infty] \times E)$, then

$$F_t^\varepsilon = F_t \exp\Big\{ - \int_0^t \int_E \varepsilon u_s(y))\lambda(s, dy) d\Lambda_s \Big\}. \tag{3.4}$$

Define

$$\psi_\varepsilon(t) = \sup\{s : F_s^\varepsilon \geq F_t\}.$$

Then $\psi_\varepsilon(t)$ is an increasing function of t, and $F_{\psi_\varepsilon(t)}^\varepsilon = F_t$, i.e.,

$$\mu^\varepsilon(]\psi_\varepsilon(t), \infty] \times E) = \mu(]t, \infty] \times E).$$

Hence if we let $\phi_\varepsilon(t) = \psi_\varepsilon^{-1}(t)$, then under μ^ε, $\phi_\varepsilon(T)$ has the same distribution as T under μ. This observation leads us to the following theorem:

THEOREM 3.1. *Let $G(t,z)$ be a real-valued function defined on $[0,\infty] \times E$, which is bounded and has bounded partial derivative $\frac{\partial}{\partial t}G(t,z)$. Then*

$$E\Big[\Big(\int_0^T \int_E u_t(y)\lambda(t,dy)q(dt,E)\Big)G(T,Z)\Big]$$

$$= -E\Big[\frac{\partial G(T,Z)}{\partial t}\ \frac{1}{\alpha(T)}\int_0^T \int_E u_t(y)\lambda(t,dy)\alpha_t dt\Big]. \tag{3.5}$$

Proof. From the above discussion we have

$$E[G(T,Z)] = E^{\varepsilon}[G(\phi_\varepsilon(T),Z)]$$

$$= E[L_T^{\varepsilon}G(\phi_\varepsilon(T),Z)] \tag{3.6}$$

where E^{ε} denotes that expectation is taken with respect to μ^{ε}. Differentiate (3.6) with respect to ε, then set $\varepsilon = 0$ to obtain

$$E\Big[\frac{dL_T^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} G(\phi_\varepsilon(T),Z)\Big|_{\varepsilon=0}\Big] + E\Big[L_T^{\varepsilon}\Big|_{\varepsilon=0}\ \frac{d}{d\varepsilon}\ G(\phi_\varepsilon(T),Z)\Big|_{\varepsilon=0}\Big] = 0. \tag{3.7}$$

From (3.2) and (3.3),

$$\frac{dL_T^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^T \int_E u_t(y)\lambda(t,dy)q(dt,E). \tag{3.8}$$

Also,

$$\frac{d}{d\varepsilon}\ G(\phi_\varepsilon(T),Z)\Big|_{\varepsilon=0} = \frac{\partial}{\partial t}\ G(T,Z)\ \frac{\partial}{\partial \varepsilon}\ \phi_\varepsilon(T)\Big|_{\varepsilon=0}.$$

To evaluate $\frac{\partial}{\partial \varepsilon}\ \phi_\varepsilon(T)\Big|_{\varepsilon=0}$, note that $F^{\varepsilon}_{\psi_\varepsilon(t)} = F_t$. Hence

$$F_t^{\varepsilon} = F_{\phi_\varepsilon(t)} = F_t \exp\Big\{-\int_0^t \int_E \varepsilon u_s(y))\lambda(s,dy)d\Lambda_s\Big\}$$

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon}\Big|_{\varepsilon=0} = F_t\Big(-\int_0^t \int_E u_s(y)\lambda(s,dy)d\Lambda_s\Big). \tag{3.9}$$

On the other hand (see Elliott (1982)),

$$F_t = \exp\Big(-\int_0^t \alpha(s)ds\Big)$$

so

$$F_{\phi_\varepsilon(t)} = \exp\Big(-\int_0^{\phi_\varepsilon(t)} \alpha(s)ds\Big).$$

Thus

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon} = -\alpha(\phi_\varepsilon(t))\,\frac{d\phi_\varepsilon(t)}{d\varepsilon}\,\exp\Big(-\int_0^{\phi_\varepsilon(t)} \alpha(s)ds\Big)$$

and

$$\frac{dF_{\phi_\varepsilon(t)}}{d\varepsilon}\Big|_{\varepsilon=0} = -\alpha(t)\,\frac{d\phi_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0} F_t. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\frac{d\phi_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{1}{\alpha(t)}\int_0^t\int_E u_s(y)\lambda(s,dy)\alpha(s)ds. \tag{3.11}$$

Now from (3.8) and (3.11), we have (3.5). □

4. Integration by Parts and Martingale Representation.

In Section 3, we considered a single jump process with values in a Lusin space. Now suppose that at its random jump time T, the process jumps to a fixed position $z_1 \in E$. If we define Λ_t^ε simply by

$$\Lambda_t^\varepsilon = \int_0^t (1+\varepsilon u_s)d\Lambda_s$$

where u is just a function of the time, which is positive, bounded and vanishes outside a bounded interval, then the method described in Section 3 would give us the simpler integration by parts formula:

$$E\Big[\Big(\int_0^T u_s dq_s\Big) G(T)\Big] = -E\Big[\frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\int_0^T u_s\alpha_s ds\Big] \tag{4.1}$$

where G is a bounded function defined on $[0,\infty]$ with bounded derivative. On the other hand, if we assume $E[G(T)] = 0$, then $G(T)$ has the martingale representation (see Elliott (1982)):

$$G(T) = \int_0^T \gamma_s dq_s \tag{4.2}$$

where

$$\gamma_s = G(s) - F_s^{-1}\int_0^s G(v)dF_v.$$

If we substitute (4.2) into the left side of (4.1), we have

$$\begin{aligned} E\Big[\Big(\int_0^T u_s dq_s\Big)\Big(\int_0^T \gamma_s dq_s\Big)\Big] &= E\Big[\int_0^T u_s\gamma_s d\langle q,q\rangle_s\Big] \\ &= -E\Big[\int_0^T u_s\gamma_s\,\frac{dF_s}{F_s}\Big] \\ &= E\Big[\int_0^T u_s\gamma_s\alpha_s ds\Big] \\ &= E\Big[\int_0^\infty I_{s\le T}\; u_s\gamma_s\alpha_s ds\Big]. \end{aligned} \tag{4.3}$$

Now, if we define the measure π by:

$$\pi(dt) = \frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\,\delta_T(dt),$$

then the right side of (4.1) is

$$\begin{aligned}-E\Big[\int_0^\infty \int_0^t u_s\alpha_s ds\mu(dt)\Big] &= -E\Big[\int_0^\infty \int_0^\infty I_{0\le s\le t<\infty}\; u_s\alpha_s ds\mu(dt)\Big]\\ &= -E\Big[\int_0^\infty \pi[s,\infty)u_s\alpha_s ds\Big]\\ &= -E\Big[\int_0^\infty \frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\; I_{s\le T<\infty}\; u_s\alpha_s ds\Big]. \qquad (4.4)\end{aligned}$$

A comparison between (4.3) and (4.4) leads us to the following expression for γ:

THEOREM 4.1. *The integrand γ that appears in the martingale representation (4.2) is given by:*

$$\gamma_s = -E\Big[\frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\; I_{s\le T<\infty} \mid \mathcal{F}_{s-}\Big]. \qquad (4.5)$$

Proof.

$$\begin{aligned}E\Big[\frac{dG(T)}{dt}\,\frac{1}{\alpha(T)}\; I_{s\le T<\infty} \mid \mathcal{F}_{s-}\Big] &= -F_s^{-1}\int_s^\infty \frac{dG(t)}{dt}\,\frac{1}{\alpha(t)}\; dF_t\\ &= F_s^{-1}\int_s^\infty \frac{dG(t)}{dt}\,\frac{1}{\alpha(t)}\; F_t\alpha(t)dt\\ &= F_s^{-1}\int_s^\infty F_t dG(t)\\ &= F_s^{-1}\Big(-F_sG(s) - \int_s^\infty G(t)dF_t\Big)\\ &= -G(s) + F_s^{-1}\int_0^s G(r)dF_r\\ &= -\gamma_s\end{aligned}$$

□

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Control of Partially Observed Diffusions[1]

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Abstract. The optimal control of a partially observed diffusion is discussed when the control parameter is present in both the drift and diffusion coefficients. Using a differentiation result of Blagovescenskii and Freidlin, and adapting techniques of Bensoussan, a stochastic minimum principle is obtained.

Key Words. Optimal control, partially observed diffusion, minimum principle.

1. Introduction

The adjoint process, and related minimum principles, for partially observed stochastic control problems have been investigated in several recent papers. See, for example, the works of Bensoussan (Ref. 1), Haussmann (Ref. 2), Baras, Elliott and Kohlmann (Ref. 3) and Elliott (Ref. 4). In these papers, however, the control variable occurs in only the drift coefficient. For a fully observed stochastic control problem Bensoussan (Ref. 5) does consider the case when the control also appears in the diffusion coefficient. This case is also discussed in (Ref. 6), and, when the optimal control is Markov, an explicit equation for the adjoint process is derived.

In this paper we consider a state process, which is only partially observed through a noisy observation process, and for which the control variable is present in both the drift and diffusion coefficients. By adapting the techniques of Bensoussan (Ref. 5) an adjoint process is described and a minimum principle obtained for an optimum control. To the best of our knowledge, this is the first paper that discusses this problem for the partially observed case when the control appears in both the drift and diffusion terms.

2. Dynamics

Suppose that the state of the system is described by a stochastic differential equation,

$$dx_t = f(t, x_t, u)dt + g(t, x_t, u)dw_t, \quad x_t \in R^d, \ x_0 = x_0, \ 0 \le t \le T. \tag{1}$$

The control parameter u will take values in a compact, convex subset U of some Euclidean space R^k.

We shall assume the following:

(A1) $x_0 \in R^d$ is given.

(A2) $f : [0,T] \times R^d \times U \to R^d$ is continuous, and continuously differentiable with respect to x, u.

(A3) $g : [0,T] \times R^d \times U \to R^d \otimes R^n$ is a continuous, matrix valued function, which is continuously differentiable with respect to x, u. The columns of g will be denoted by $g^{(k)}$ for $k = 1, 2, \ldots, n$.

(A4) There is a constant K such that

$$(1+|x|)^{-1}|f(t,x,u)| + |f_x(t,x,u)| + |f_u(t,x,u)| \leq K$$

$$|g(t,x,u)| + |g_x(t,x,u)| + |g_u(t,x,u)| \leq K.$$

Suppose the observation process is given by

$$dy_t = h(x_t)dt + dv_t, \quad y_t \in R^m, \ y_0 = 0, \ 0 \leq t \leq T. \tag{2}$$

In the above equations $w = (w^1, \ldots, w^n)$ and $v = (v^1, \ldots, v^m)$ are independent Brownian motions. We also assume:

(A5) $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x, and for some constant K_1,

$$|h(x)| + |h_x(x)| \leq K_1.$$

Let $\widehat{P}$ denote Wiener measure on $C([0,T], R^n)$ and μ denote Wiener measure on $C([0,T], R^m)$. Consider the space $\Omega = C([0,T], R^n) \times C([0,T], R^m)$ with coordinate functions (w_t, y_t) and define Wiener measure P on Ω by

$$P(dw, dy) = \widehat{P}(dw)\mu(dy).$$

Definition 2.1. Write $\{F_t\}$ for the right continuous, complete filtration on $C([0,T],R^n)$ generated by $F_t^0 = \sigma\{w_s,\ s \le t\}$. Write $Y = \{Y_t\}$ for the right continuous complete filtration on $C([0,T],R^m)$ generated by $Y_t^0 = \sigma\{y_s - y_r,\ 0 \le r \le s \le t\}$. The set of admissible control functions $\underline{U}$ will be the Y-predictable functions on $[0,T] \times C([0,T],\ R^m)$ with values in U. Then

$$\begin{aligned}\underline{U} &\subset L^2_Y[0,T] \\ &= \{v(t,w') : v(t,w') \in L^2([0,T] \times (C([0,T],R^m)),\ dt \times d\mu;\ R^k), \\ &\qquad \text{for a.e. } t,\ v(t,\cdot) \in L^2(C([0,T],R^m), Y_t, d\mu, R^k)\}.\end{aligned}$$

For $u \in \underline{U}$, write $X^u_{s,t}(x)$ for the unique strong solution of (1) corresponding to control u, and with $X^u_{s,s}(x) = x$.

Write

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(X^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t |h(X^u_{s,r}(x))|^2 dr\right) \tag{3}$$

and define a new probability measure P^u on Ω by

$$\frac{dP^u}{dP} = Z^u_{0,T}(x_0).$$

Then under P^u, $(X^u_{0,t}(x_0),\ y_t)$ is a solution of (1) and (2).

Cost. We shall suppose the cost is

$$C(X^u_{0,T}(x_0)) + \int_0^T \ell(r, X^u_{0,r}(x_0), u_r) dr.$$

We suppose

(A6) $|C(x)| + |C_x(x)| + |C_{xx}(x)| \le K(1 + |x|^q)$, for some $q < \infty$.

(A7) $\ell : [0,T] \times R^d \times U \to R$ is Borel measurable and continuously differentiable in (x,u). Furthermore ℓ and its derivatives in x and u satisfy linear growth conditions in x.

The expected cost if a control $u \in \underline{U}$ is used is, therefore,

$$J(u) = E^u\Big[C(X^u_{0,T}(x_0)) + \int_0^T \ell(r, X^u_{0,r}(x_0), u_r)dr\Big].$$

In terms of P, this is

$$J(u) = E\Big[Z^u_{0,T}(x_0)\Big(C(X^u_{0,T}(x_0)) + \int_0^T \ell(r, X^u_{0,r}(x_0), u_r)dr\Big)\Big].$$

Consider the $d+1$ dimensional system given by

$$\begin{aligned} X^u_{s,t} &= x + \int_s^t f(r, X^u_{s,r}(x), u_r)dr + \int_s^t g(r, X^u_{s,r}(x), u_r)dW_r \\ Z^u_{s,t} &= z + \int_s^t Z^u_{s,r} h(X^u_{s,r}(x))dy_r. \end{aligned} \tag{4}$$

Write

$$\widetilde{X}^u_{s,t} = \begin{pmatrix} X^u_{s,t} \\ Z^u_{s,t} \end{pmatrix} \qquad \tilde{f}(r) = \begin{pmatrix} f(r, X^u_{s,r}(x), u_r) \\ 0 \end{pmatrix}$$

$$\tilde{g}(r) = \begin{pmatrix} g(r, X^u_{s,r}(x), u_r) & 0 \\ 0 & Z^u_{s,r}(x,z)h(X^u_{s,r}(x)) \end{pmatrix}$$

$$\widetilde{W}_r = \begin{pmatrix} w_r \\ y_r \end{pmatrix} \qquad \widetilde{X} = \begin{pmatrix} x \\ z \end{pmatrix}.$$

Then we can write (4) as

$$\widetilde{X}^u_{s,t}(\tilde{x}) = \tilde{x} + \int_s^t \tilde{f}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)dr + \int_s^t \tilde{g}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)d\widetilde{W}_r. \tag{5}$$

As in (Ref. 3) we can assume the Jacobian $\dfrac{\partial \widetilde{X}^u_{s,t}(\tilde{x})}{\partial \tilde{x}} = \widetilde{D}^u_{s,t}$ exists for all s, t, $\tilde{x}$ and all w not in a set of measure zero, and is the solution of

$$\widetilde{D}^u_{s,t} = I + \int_s^t \tilde{f}_{\tilde{x}}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)\widetilde{D}^u_{s,r}dr + \sum_{i=1}^{n+m} \int_s^t \tilde{g}^{(i)}_{\tilde{x}}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)\widetilde{D}^u_{s,r}d\widetilde{W}^i_r. \quad (6)$$

Here I is $(d+1) \times (d+1)$ identity matrix. In fact, if the coefficients $\tilde{f}$ and $\tilde{g}$ are C^k the map $\tilde{x} \to \widetilde{X}^u_{s,t}(\tilde{x})$ is C^{k-1}.

Similarly to (Ref. 3) the matrix process $\widetilde{H}$ defined by

$$\widetilde{H}^u_{s,t} = I - \int_s^t \widetilde{H}^u_{s,r}\Big(\tilde{f}_{\tilde{x}}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r) - \sum_{k=1}^{n+m} \tilde{g}^{(k)}_{\tilde{x}}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)^2\Big)dr$$

$$- \sum_{k=1}^{m+n} \int_s^t \widetilde{H}^u_{s,r}\tilde{g}^{(k)}_{\tilde{x}}(r, \widetilde{X}^u_{s,r}(\tilde{x}), u_r)d\widetilde{W}^k_r. \quad (7)$$

exists and $\widetilde{H}^u_{s,t} = (\widetilde{D}^u_{s,t})^{-1}$.

Remark 2.1. Write $\|\widetilde{X}^u(\tilde{x}_0)\|_t = \sup_{0\le s\le t} |\widetilde{X}^u_{0,s}(\tilde{x}_0)|$, $\|\widetilde{D}^u\|_t = \sup_{0\le s\le t} |\widetilde{D}^u_{0,s}|$, $\|\widetilde{H}^u\|_t = \sup_{0\le s\le t} |\widetilde{H}^u_{0,s}|$. Then $\|\widetilde{X}^u(\tilde{x}_0)\|_T$, $\|\widetilde{D}^u\|_T$, $\|\widetilde{H}^u\|_T$ are in L^p, $1 \le P < \infty$.

We shall suppose there is an optimal control $u^* \in \underline{U}$, so that $J(u^*) \le J(u)$ for all other $u \in \underline{U}$.

Notation 2.1. We shall write $\widetilde{X}^*$ for $\widetilde{X}^{u^*}$ and $\widetilde{D}^*_{0,t}$ for $\widetilde{D}^{u^*}_{0,t}$, etc.

3. Differentiability

Suppose $u^* \in \underline{U}$ is an optimal control. Consider any other control $v \in \underline{U}$. Then for $\theta \in [0,1]$

$$u_\theta(t) = u^*(t) + \theta(v(t) - u^*(t)) \in \underline{U}$$

and

$$J(u_\theta) \ge J(u^*). \quad (8)$$

If the Gâteaux derivative $J'(u^*)$ of J, as a functional on the Hilbert space $L^2_Y[0,T]$, is well defined, differentiating (8) in θ implies

$$\langle J'(u^*),\ v(t) - u^*(t)\rangle \geq 0$$

for all $v \in \underline{U}$.

LEMMA 3.1. *Suppose $v \in \underline{U}$ is such that $u^*_\theta = u^* + \theta v \in \underline{U}$ for $\theta \in [0,\alpha]$. Write $\widetilde{X}^\theta_{0,t}(\tilde{x}_0)$ for the trajectory associated with u^*_θ. Then $M_t = \left.\dfrac{\partial \widetilde{X}^\theta_{0,t}(\tilde{x}_0)}{\partial\theta}\right|_{\theta=0}$ exists a.s. and is the unique solution of the equation*

$$\begin{aligned} M_t = &\int_0^t (\tilde{f}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)M_r + \tilde{f}_u(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)v_r)dr \\ &+ \sum_{i=1}^{n+m} \int_0^t \tilde{g}^{(i)}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)M_r d\widetilde{W}^i_r \\ &+ \sum_{i=1}^{n} \int_0^t \tilde{g}^{(i)}_{u}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)v_r d\widetilde{W}^i_r, \end{aligned} \tag{9}$$

because for $n+1 \leq i \leq n+m$, $\tilde{g}^{(i)}_u = 0$.

Proof. The result follows from the theorem of Blagovescenskii and Freidlin (Refs. 7–8) on the differentiability of solutions of stochastic differential equations which depend on a parameter. In effect the result of (Refs. 7–8) states that, if the coefficients are differentiable, the equation for the derivative is obtained by differentiation. Considering the initial condition as a parameter this result gives, in particular, the equation for the differential or Jacobian as in (6). □

LEMMA 3.2. *Write*

$$\begin{aligned} \tilde{\eta}_{0,t} = &\int_0^t (\widetilde{D}^*_{0,r})^{-1}\tilde{f}_u(r)v_r dr + \sum_{i=1}^{n}\int_0^t (\widetilde{D}^*_{0,r})^{-1}\tilde{g}^{(i)}_u(r)v_r d\widetilde{W}^i_r \\ &- \sum_{i=1}^{n}\int_0^t (\widetilde{D}^*_{0,r})^{-1}\tilde{g}^{(i)}_{\tilde{x}}(r)\tilde{g}^{(i)}_u(r)v_r dr \end{aligned} \tag{10}$$

where $\tilde{f}_u$, $\tilde{g}_{\tilde{x}}$, $\tilde{g}_u$ *are as in equation (9). Then* $M_t = \widetilde{D}^*_{0,t}\tilde{\eta}_{0,t}$.

Proof. By differentiating, we see the product $\widetilde{D}^*_{0,t}\tilde{\eta}_{0,t}$ satisfies equation (9). □

LEMMA 3.3.

$$\frac{dJ(u^*_\theta)}{d\theta}\bigg|_{\theta=0} = E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))\widetilde{D}^*_{0,T}\tilde{\eta}_{0,T} + \int_0^T (\tilde{\ell}_{\tilde{x}}(r,\widetilde{X}^*_{0,r}(\tilde{x}_0),u^*_r)\widetilde{D}^*_{0,r}\tilde{\eta}_{0,r} + \tilde{\ell}_u(r,\widetilde{X}^*_{0,r}(x_0),u^*_r)v_r)dr\Big]$$

where

$$\widetilde{C}(\widetilde{X}^*_{0,T}(\tilde{x}_0)) = Z^*_{0,T}(x_0)C(X^*_{0,T}(x_0))$$

$$\tilde{\ell}(r,\widetilde{X}^*_{0,r},u^*_r) = Z^*_{0,r}(x_0)\ell(r,X^*_{0,r}(x_0),u^*_r).$$

Proof.

$$\begin{aligned} J(u^*_\theta) &= E\Big[\widetilde{C}(\widetilde{X}^\theta_{0,T}(\tilde{x}_0)) + Z^\theta_{0,T}(x_0)\int_0^T \ell(r,X^\theta_{0,r}(x_0),u^*_\theta(r))dr)\Big] \\ &= E\Big[\widetilde{C}(\widetilde{X}^\theta_{0,T}(\tilde{x}_0)) + \int_0^T Z^\theta_{0,r}\ell(r,X^\theta_{0,r}(x_0),u^*_\theta(r))dr + \int_0^T\Big(\int_0^r \ell(s)ds\Big)dZ^\theta_{0,r}\Big] \\ &= E\Big[\widetilde{C}(\widetilde{X}^\theta_{0,T}(\tilde{x}_0)) + \int_0^T \tilde{\ell}(r,\widetilde{X}^\theta_{0,r}(\tilde{x}_0),u^*_\theta(r))dr\Big]. \end{aligned}$$

So

$$\frac{dJ(u^*_\theta)}{d\theta}\bigg|_{\theta=0} = E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))M_T + \int_0^T (\tilde{\ell}_{\tilde{x}}(r,\widetilde{X}^*_{0,r}(\tilde{x}_0),u^*_r)M_r + \tilde{\ell}_u(r,\widetilde{X}^*_{0,r}(\tilde{x}_0),u^*_r)v_r)dr\Big]$$

substituting $M_t = \widetilde{D}^*_{0,t}\tilde{\eta}_{0,t}$ the result follows. □

Consider the right continuous version of the square integrable martingale

$$N_t = E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))\widetilde{D}^*_{0,T} + \int_0^T \tilde{\ell}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)\widetilde{D}^*_{0,r}dr \mid \mathcal{G}_t\Big]$$

where $\mathcal{G}_t$ is the right continuous complete σ-field on Ω, generated by $\mathcal{G}^0_t = F^0_t \otimes Y^0_t$.

From (Ref. 9) N_t has a martingale representation

$$N_t = E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))\widetilde{D}^*_{0,T} + \int_0^T \tilde{\ell}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)\widetilde{D}^*_{0,r}dr\Big] + \sum_{i=1}^{n+m} \int_0^t \tilde{\gamma}^i_r d\widetilde{W}^i_r$$

where the $\tilde{\gamma}^i_r$ are $\mathcal{G}_r$ predictable processes such that

$$E\Big[\int_0^T (\tilde{\gamma}^i_r)^2 dr\Big] < \infty.$$

Write

$$\begin{aligned}
\xi_t &= N_t - \int_0^t \tilde{\ell}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)\widetilde{D}^*_{0,r}dr \\
\tilde{p}_t &= \xi_t(\widetilde{D}^*_{0,t})^{-1} \\
&= E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))\widetilde{D}^*_{t,T} + \int_t^T \tilde{\ell}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)\widetilde{D}^*_{t,r}dr \mid \mathcal{G}_t\Big].
\end{aligned}$$

THEOREM 3.1.

$$\begin{aligned}
\frac{dJ(u^{\tilde{\ }}_\theta)}{d\theta}\Big|_{\theta=0} = E\Big[\int_0^T \tilde{p}_s\tilde{f}_u(s)v_s ds - \sum_{i=1}^n \int_0^T \tilde{p}_s\tilde{g}^{(i)}_{\tilde{x}}(s)\tilde{g}^{(i)}_u(s)v_s ds + \int_0^T \tilde{\ell}_u(s)v_s ds \\
+ \sum_{i=1}^n \int_0^T \tilde{\gamma}^i_s(\widetilde{D}^*_{0,s})^{-1}\tilde{g}^{(i)}_u(s)v_s ds\Big]. \qquad (11)
\end{aligned}$$

Proof. The product rule gives

$$\begin{aligned}\xi_T \cdot \tilde{\eta}_{0,T} = &\int_0^T \xi_r (\widetilde{D}^*_{0,s})^{-1} \tilde{f}_u(s) v_s ds \\ &+ \sum_{i=1}^n \int_0^T \xi_s (\widetilde{D}^*_{0,s})^{-1} \tilde{g}_u^{(i)}(s) v_s d\widetilde{W}_s^i - \sum_{i=1}^n \int_0^T \xi_s (\widetilde{D}^*_{0,s})^{-1} g_x^{(i)}(s) g_u^{(i)}(s) v_s ds \\ &+ \sum_{i=1}^{n+m} \int_0^T \tilde{\gamma}_s^{(i)} \tilde{\eta}_{0,s} d\widetilde{W}_s^{(i)} - \int_0^T \tilde{\ell}_{\tilde{x}}(s) (\widetilde{D}^*_{0,s})^{-1} \tilde{\eta}_{0,s} ds \\ &+ \sum_{i=1}^n \int_0^T (\widetilde{D}^*_{0,s})^{-1} g_u^{(i)}(s) v_s \tilde{\gamma}_s^{(i)} ds. \end{aligned} \tag{12}$$

However, from Lemma 3.3

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = E\Big[\xi_T \cdot \tilde{\eta}_{0,T} + \int_0^T (\tilde{\ell}_{\tilde{x}}(s) \widetilde{D}^*_{0,s} \tilde{\eta}_{0,s} + \tilde{\ell}_u(s) v_s) ds\Big]. \tag{13}$$

Substituting (12) in (13) and using the definition of $\tilde{p}$, the result follows. □

Remark 3.1. Write $\widetilde{X}^*_{t,T}(\tilde{x}) = (X^*_{t,T}(x), Z^*_{t,T}(x,z))'$ for the solution of (4) using control u^*. Then, by uniqueness,

$$Z^*_{t,T}(x,z) = z Z^*_{t,T}(x,1) \tag{14}$$

and $Z^*_{t,T}(x,1)$ is the density given by (3).

LEMMA 3.4.

$$\frac{\partial Z^*_{t,T}(x,z)}{\partial z} = Z^*_{t,T}(x,1) \tag{15}$$

$$= Z^{*-1}_{0,t}(x_0,1) Z^*_{0,T}(x,1) \tag{16}$$

and

$$\frac{\partial Z^*_{t,T}(x,1)}{\partial x} = Z^*_{t,T}(x,1)\Big(\int_0^T \frac{\partial h(x^*_{t,r})}{\partial x} \cdot D^*_{t,r} dv_r\Big) \tag{17}$$

where $D^*_{t,r} = \dfrac{\partial X^*_{t,r}}{\partial x}$.

Proof. (15) is immediate from (14). Now

$$Z^*_{t,T}(x,1) = 1 + \int_t^T Z^*_{t,r}(x,1)h(X^*_{t,r}(x))dy_r.$$

Applying the differentiation result of Blagovescenskii and Freidlin (Ref. 7–8) we have

$$\frac{\partial Z^*_{t,T}(x,1)}{\partial x} = \int_t^T \frac{\partial Z^*_{t,r}(x,1)}{\partial x} h(X^*_{t,r}(x))dy_r + \int_t^T Z^*_{t,r}(x,1)\ \frac{\partial h}{\partial x}(X^*_{t,r}(x))D^*_{t,r}dy_r.$$

This equation can be solved by variation of constants to give

$$\frac{\partial Z^*_{t,T}(x,1)}{\partial x} = Z^*_{t,T}(x,1)\Big(\int_t^T \frac{\partial h}{\partial x}(X^*_{t,r}(x))D^*_{t,r}dy_r - \int_t^T \frac{\partial h}{\partial x}(X^*_{t,r}(x))D^*_{t,r}\cdot h(X^*_{t,r}(x))dr\Big)$$

and the result follows from (2). □

Notation 3.1. Write $Z^*_{0,t}$ for $Z^*_{0,t}(x_0,1)$, $Z^*_{t,T}$ for $Z^*_{t,T}(x,1)$,

$$\phi(t) = \Big(C_x(X^*_{0,T}(x_0))D^*_{t,T} + C(X^*_{0,T}(x_0))\Big(\int_t^T \frac{\partial h}{\partial x}\cdot D^*_{t,r}dv_r\Big), Z^{*-1}_{0,t}C(X^*_{0,T}(x_0))\Big)$$

and

$$\psi(r) = \Big(\ell_x(r)D^*_{t,r} + \ell(r)\Big(\int_t^r \frac{\partial h}{\partial x}\ D^*_{t,\eta}\cdot dv_\eta\Big), Z^{*-1}_{0,t}\ell(r)\Big).$$

Note that the linear growth conditions of ℓ and ℓ_x, the integrability properties of D^* and the boundedness of h and h_x imply that

$$\int_t^s \Big(\int_t^r \psi(\eta)d\eta\Big)dZ^*_{t,r}$$

is a square integrable martingale.

LEMMA 3.6.

$$\tilde{p}_t = E^*\Big[Z^*_{0,t}\Big(\phi(t) + \int_t^T \psi(r)dr\Big) \mid \mathcal{G}_t\Big]. \tag{18}$$

Proof.

$$\begin{aligned}
\tilde{p}_t &= E\Big[\widetilde{C}_{\tilde{x}}(\widetilde{X}^*_{0,T}(\tilde{x}_0))\widetilde{D}^*_{t,T} + \int_t^T \tilde{\ell}_{\tilde{x}}(r, \widetilde{X}^*_{0,r}(\tilde{x}_0), u^*_r)\widetilde{D}^*_{t,r}dr \mid \mathcal{G}_t\Big] \\
&= E\Big[\Big(Z^*_{0,T}C_x(X^*_{0,T}(x_0))D^*_{t,T} + Z^*_{0,t}\frac{\partial Z^*_{t,T}(x)}{\partial x}C(X^*_{0,T}(x_0)), Z^*_{t,T}C(X^*_{0,T}(x_0))\Big) \\
&+ \int_t^T \Big(Z^*_{0,r}\ell_x(r)D^*_{t,r} + Z^*_{0,r}\,\frac{\partial Z^*_{t,r}(x)}{\partial x}\cdot \ell(r), Z^*_{t,r}\ell(r)\Big)dr \mid \mathcal{G}_t\Big].
\end{aligned}$$

Substituting (17) this is

$$\begin{aligned}
&= E\Big[Z^*_{0,T}\Big\{C_x(X^*_{0,T}(x_0))D^*_{t,T} + C(X^*_{0,T}(x_0)\Big(\int_t^T \frac{\partial h}{\partial x}\cdot D^*_{t,r}dv_r\Big), Z^{*-1}_{0,t}C(X^*_{0,T}(x_0))\Big\} \\
&+ \int_t^T Z^*_{0,r}\Big\{\ell_x(r)D^*_{t,r} + \ell(r)\Big(\int_t^r \frac{\partial h}{\partial x}\cdot D^*_{t,\eta}\cdot dv_\eta\Big), Z^{*-1}_{0,t}\ell(r)\Big\}dr \mid \mathcal{G}_t\Big] \\
&= E\Big[Z^*_{0,T}\phi(t) + \int_t^T Z^*_{0,r}\psi(r)dr \mid \mathcal{G}_t\Big].
\end{aligned} \tag{19}$$

Now

$$\begin{aligned}
E\Big[\int_t^T Z^*_{0,r}\psi(r)dr \mid \mathcal{G}_t\Big] &= Z^*_{0,t}E\Big[\int_t^T Z^*_{t,r}\psi(r)dr \mid \mathcal{G}_t\Big] \\
&= Z^*_{0,t}E\Big[Z^*_{t,T}\int_t^T \psi(r)dr - \int_t^T\Big(\int_t^r \psi(\eta)d\eta\Big)dZ^*_{t,r} \mid \mathcal{G}_t\Big].
\end{aligned}$$

However, the last term is a square integrable $(P, \mathcal{G}_t)$ martingale, so

$$\begin{aligned}
E\Big[\int_t^T Z^*_{0,r}\psi(r)dr \mid \mathcal{G}_t\Big] &= Z^*_{0,t}E\Big[Z^*_{t,T}\int_t^T \psi(r)dr \mid \mathcal{G}_t\Big] \\
&= E\Big[Z^*_{0,T}\int_t^T \psi(r)dr \mid \mathcal{G}_t\Big].
\end{aligned}$$

Substituting in (19),

$$\tilde{p}_t = E\Big[Z^*_{0,T}\Big(\phi(t) + \int_t^T \psi(r)dr\Big) \mid \mathcal{G}_t\Big]$$

and using Bayes' formula, this is

$$= E^*\Big[\phi(t) + \int_t^T \psi(r)dr \mid \mathcal{G}_t\Big] Z^*_{0,t}$$

$$= E^*\Big[Z^*_{0,t}\Big(\phi(t) + \int_t^T \psi(r)dr\Big) \mid \mathcal{G}_t\Big].$$

□

Definition 3.2. The adjoint process will be the process p defined by

$$p_s = E[\tilde{p}_s \mid Y_s \vee \{x\}] \frac{E[Z^*_{0,s} \mid Y_s]}{E[Z^*_{0,s} \mid Y_s \vee \{x\}]}$$

$$= E\Big[Z^*_{0,T}\Big(\phi(s) + \int_s^T \psi(r)dr\Big) \Big| Y_s \vee \{x\}\Big] \frac{E[Z^*_{0,s} \mid Y_s]}{E[Z^*_{0,s} \mid Y_s \vee \{x\}]}$$

$$= E^*\Big[\phi(s) + \int_s^T \psi(r)dr \Big| Y_s \vee \{x\}\Big] E[Z^*_{0,s} \mid Y_s]$$

$$= E^*\Big[\Big(\phi(s) + \int_s^T \psi(r)dr\Big) E[Z^*_{0,s} \mid Y_s] \Big| Y_s \vee \{x\}\Big].$$

As in Bensoussan (Ref. 1), the adjoint process depends on x, which represents the state of the process at time s. However, x is just a parameter which is integrated out in the minimum principle of Theorem 3.2.

Returning to the perturbation

$$u_\theta(t) = u^*(t) + \theta(v(t) - u^*(t))$$

of the optimal control, we have

$$\frac{dJ(u_\theta)}{d\theta}\Big|_{\theta=0} \geq 0.$$

That is

$$E\Big[\int_0^T \tilde{p}_s\tilde{f}_u(s)(v(s)-u^*(s))ds - \sum_{i=1}^n \int_0^T \tilde{p}_s\tilde{g}_{\tilde{x}}^{(i)}(s)\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))ds$$
$$+\int_0^T \tilde{\ell}_u(s)(v(s)-u^*(s))ds + \sum_{i=1}^n \int_0^T \tilde{\gamma}_s^i(\widetilde{D}_{0,s}^*)^{-1}\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))ds\Big] \geq 0$$

for all $v \in \underline{U}$. Now

$$E\Big[\int_0^T \tilde{p}_s\tilde{f}_u(s)(v(s)-u^*(s))ds - \sum_{i=1}^n \int_0^T \tilde{p}_s\tilde{g}_{\tilde{x}}^{(i)}(s)\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))ds$$
$$+\int_0^T \tilde{\ell}_u(s)(v(s)-u^*(s))ds + \sum_{i=1}^n \int_0^T \tilde{\gamma}_s^i(\widetilde{D}_{0,s}^*)^{-1}\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))ds\Big]$$
$$= E\Big[\int_0^T E\Big[\tilde{p}_s\tilde{f}_u(s) - \sum_{i=1}^n \tilde{p}_s\tilde{g}_{\tilde{x}}^{(i)}(s)\tilde{g}_u^{(i)}(s) + \tilde{\ell}_u(s) + \sum_{i=1}^n \tilde{\gamma}_s^i(\widetilde{D}_{0,s}^*)^{-1}\tilde{g}_u^{(i)}(s) \mid Y_s\Big]$$
$$\cdot (v(s)-u^*(s))ds\Big] \geq 0. \tag{20}$$

Therefore, because (20) is true for all $v \in \underline{U}$, we have for a.e. t and a.s. w.

$$E\Big[\tilde{p}_s\tilde{f}_u(s)(v(s)-u^*(s)) - \sum_{i=1}^n \tilde{p}_s\tilde{g}_{\tilde{x}}^{(i)}(s)\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))$$
$$+\tilde{\ell}_u(s)(v(s)-u^*(s)) + \sum_{i=1}^n \tilde{\gamma}_s^i(\widetilde{D}_{0,s}^*)^{-1}\tilde{g}_u^{(i)}(s)(v(s)-u^*(s)) \,\Big|\, Y_s\Big] \geq 0 \tag{21}$$

for all $v \in \underline{U}$.

From (21)

$$E\Big[\tilde{p}_s\tilde{f}_u(s)(v(s)-u^*(s)) - \sum_{i=1}^n \tilde{p}_s\tilde{g}_{\tilde{x}}^{(i)}(s)\tilde{g}_u^{(i)}(s)(v(s)-u^*(s))$$
$$+\tilde{\ell}_u(s)(v(s)-u^*(s)) + \sum_{i=1}^n \tilde{\gamma}_s^i(\widetilde{D}_{0,s}^*)^{-1}\tilde{g}_u^{(i)}(s)(v(s)-u^*(s)) \,\Big|\, Y_s\Big]$$

$$
\begin{aligned}
&= E\Big[E\Big[Z^*_{0,T}(\phi(s) + \int_s^T \psi(r)dr) \Bigm| \mathcal{G}_s\Big]\tilde{f}_u(s)(v(s) - u^*(s)) \\
&\quad - \sum_{i=1}^n E\Big[Z^*_{0,T}(\phi(s) + \int_s^T \psi(r)dr) \Bigm| \mathcal{G}_s\Big]\tilde{g}^{(i)}_{\tilde{x}}(s)\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \\
&\quad + Z^*_{0,T}(Z^{*-1}_{0,T}\tilde{\ell}_u(s))(v(s) - u^*(s)) \\
&\quad + Z^*_{0,T}\sum_{i=1}^n Z^{*-1}_{0,T}\tilde{\gamma}^i_s(\widetilde{D}^*_{0,s})^{-1}\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \Bigm| Y_s\Big] \\
&= E^*\Big[\Big(\phi(s) + \int_s^T \psi(r)dr\Big)\tilde{f}_u(s)(v(s) - u^*(s)) \\
&\quad - \sum_{i=1}^n \Big(\phi(s) + \int_s^T \psi(r)dr\Big)\tilde{g}^{(i)}_{\tilde{x}}(s)\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \\
&\quad + Z^{*-1}_{0,T}\tilde{\ell}_u(s)(v(s) - u^*(s)) \\
&\quad + \sum_{i=1}^n Z^{*-1}_{0,T}\tilde{\gamma}^i_s(\widetilde{D}^*_{0,s})^{-1}\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \Bigm| Y_s\Big] \cdot E\Big[Z^*_{0,s} \Bigm| Y_s\Big] \\
&= E^*\Big[\Big(\phi(s) + \int_s^T \psi(r)dr\Big)E[Z^*_{0,s} \mid Y_s]\tilde{f}_u(s)(v(s) - u^*(s)) \\
&\quad - \sum_{i=1}^n \Big(\phi(s) + \int_s^T \psi(r)dr\Big)E[Z^*_{0,s} \mid Y_s]\tilde{g}^{(i)}_{\tilde{x}}(s)\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \\
&\quad + Z^{*-1}_{0,T}\tilde{\ell}_u(s)E[Z^*_{0,s} \mid Y_s](v(s) - u^*(s)) \\
&\quad + \sum_{i=1}^n Z^{*-1}_{0,T}\tilde{\gamma}^i_s(\widetilde{D}^*_{0,s})^{-1}E[Z^*_{0,s} \mid Y_s]\tilde{g}^{(i)}_u(s)(v(s) - u^*(s)) \mid Y_s\Big] \geq 0. \qquad (22)
\end{aligned}
$$

Write

$$
\tilde{\tilde{\ell}}(s) = E[\tilde{\ell}(s) \mid Y_s \vee \{x\}]\,\frac{E[Z^*_{0,s} \mid Y_s]}{E[Z^*_{0,s} \mid Y_s \vee \{x\}]} = E^*[\ell(s) \mid Y_s \vee \{x\}]E[Z^*_{0,s} \mid Y_s]
$$

$$
\gamma^i_s = E[\tilde{\gamma}^i_s(D^*_{0,s})^{-1} \mid Y_s \vee \{x\}]\,\frac{E[Z^*_{0,s} \mid Y_s]}{E[Z^*_{0,s} \mid Y_s \vee \{x\}]}.
$$

Define the Hamiltonian by

$$H(\tilde{x}, v, t, p(t)) = p_t \tilde{f}_u(t, \tilde{x}, v) - \sum_{i=1}^{n} p_t \tilde{g}_{\tilde{x}}^{(i)}(t, \tilde{x}, u_t^*) \tilde{g}(t, \tilde{x}, v) + \tilde{\tilde{\ell}}(t, \tilde{x}, v) + \sum_{i=1}^{n} \gamma_t^i \tilde{g}^{(i)}(t, \tilde{x}, v).$$

THEOREM 3.2. *If u^* is the optimal control, then a.e. s*

$$E^*\Big[\frac{\partial H}{\partial v}(\tilde{x}, u^*, s, p(s))(v(s) - u^*(s)) \,\Big|\, Y_s\Big] \geq 0 \qquad \text{a.s.}$$

Proof. From (14), $\tilde{f}_u(s)$ and $\tilde{g}^{(i)}(s)$ $(i \leq n)$ are $Y_s \vee \{x\}$ measurable. Therefore,

$$\begin{aligned}
0 \leq E^*\Big[&\Big(\phi(s) + \int_s^T \psi(r)dr\Big) E[Z_{0,s}^* \mid Y_s] \tilde{f}_u(s)(v(s) - u^*(s)) \\
&- \sum_{i=1}^{n} \Big(\phi(s) + \int_s^T \psi(r)dr\Big) E[Z_{0,s}^* \mid Y_s] \tilde{g}_{\tilde{x}}^{(i)}(s) \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \\
&+ Z_{0,T}^{*-1} \tilde{\ell}_u(s) E[Z_{0,s}^* \mid Y_s](v(s) - u^*(s)) \\
&+ \sum_{i=1}^{n} Z_{0,T}^{*-1} \tilde{\gamma}_s^i (\widetilde{D}_{0,s}^*)^{-1} E[Z_{0,s}^* \mid Y_s] \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \,\Big|\, Y_s\Big] \\
= E^*\Big\{E^*\Big[&\Big(\phi(s) + \int_s^T \psi(r)dr\Big) E[Z_{0,s}^* \mid Y_s] \tilde{f}_u(s)(v(s) - u^*(s)) \\
&- \sum_{i=1}^{n} \Big(\phi(s) + \int_s^T \psi(r)dr\Big) E[Z_{0,s}^* \mid Y_s] \tilde{g}_{\tilde{x}}^{(i)}(s) \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \\
&+ Z_{0,T}^{*-1} \tilde{\ell}_u(s) E[Z_{0,s}^* \mid Y_s](v(s) - u^*(s)) \\
&+ \sum_{i=1}^{n} Z_{0,T}^{*-1} \tilde{\gamma}_s^i (\widetilde{D}_{0,s}^*)^{-1} E[Z_{0,s}^* \mid Y_s] \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \,\Big|\, Y_s \vee \{x\}\Big] \,\Big|\, Y_s\Big\} \\
= E^*\Big[&p_s \tilde{f}_u(s)(v(s) - u^*(s)) - \sum_{i=1}^{n} p_s \tilde{g}_{\tilde{x}}^{(i)}(s) \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \\
&+ E^*[Z_{0,T}^{*-1} \tilde{\ell}_u(s) \mid Y_s \vee \{x\}] \cdot E[Z_{0,s}^* \mid Y_s](v(s) - u^*(s)) \\
&+ \sum_{i=1}^{n} E^*[Z_{0,T}^{*-1} \tilde{\gamma}_s^i (\widetilde{D}_{0,s}^*)^{-1} \mid Y_s \vee \{x\}] E[Z_{0,s}^* \mid Y_s] \tilde{g}_u^{(i)}(s)(v(s) - u^*(s)) \,\Big|\, Y_s\Big]
\end{aligned}$$

$$= E^*\Big[p_s\tilde f_u(s)(v(s)-u^*(s)) - \sum_{i=1}^n p_s\tilde g_{\tilde x}^{(i)}(s)\tilde g_u^{(i)}(s)(v(s)-u^*(s))$$

$$+ E[\tilde\ell_u(s) \mid Y_s \vee \{x\}]\ \frac{E[Z_{0,s}^* \mid Y_s]}{E[Z_{0,s}^* \mid Y_s \vee \{x\}]}\cdot(v(s)-u^*(s))$$

$$+ \sum_{i=1}^n E[\tilde\gamma_s^i(\widetilde D_{0,s}^*)^{-1} \mid Y_s \vee \{x\}]\ \frac{E[Z_{0,s}^* \mid Y_s]}{E[Z_{0,s}^* \mid Y_s \vee \{x\}]}\cdot \tilde g_u^{(i)}(s)(v(s)-u^*(s))\ \Big|\ Y_s\Big]$$

$$= E^*\Big[p_s\tilde f_u(s)(v(s)-u^*(s)) - \sum_{i=1}^n p_s\tilde g_{\tilde x}^{(i)}(s)\tilde g_u^{(i)}(s)(v(s)-u^*(s))$$

$$+ \tilde{\tilde\ell}_u(s)(v(s)-u^*(s)) + \sum_{i=1}^n \gamma_s^i\tilde g_u^{(i)}(s)(v(s)-u^*(s)) \mid Y_s\Big]$$

$$= E^*\Big[\frac{\partial H}{\partial v}(\tilde x, u^*, s, p(s))(v(s)-u^*(s))\ \Big|\ Y_s\Big].$$

So the result follows. $\square$

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Filtering and Estimation of a Markov Chain

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Abstract: A finite state Markov chain is considered. Only certain of its jumps, or alternatively, only the total number of its jumps, are observed. Based on this information a recursive estimate for the state of the chain is derived. The novel features are the representation of certain basic martingales associated with the Markov chain, and the consequent use of martingale calculus and a product technique, which simplify related formulae and calculations in the book of Brémaud. The Zakai equation is obtained and a related control problem presented in separated form.

0. Introduction

A finite state, continuous time Markov chain is considered. The state space is taken to be, without loss of generality, the set of unit vectors $S = \{e_i\}$, $e_i = (0, 0, \ldots, 1, \ldots, 0)^*$ of R^{N+1}, thus facilitating the use of linear algebra. Some basic martingales associated with the chain are identified and natural filtering problems discussed. These consider, for example, the estimation of the state of the Markov chain if only the total number of jumps, or the number of jumps into certain states, are observed. Such formulae can be obtained by specializing a general result in the book of Brémaud [1]; however, using the

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basic martingales and a product technique, our proofs are different and direct. The related Zakai equation for the unnormalized conditional distribution is then obtained. It is shown how this can be used to discuss in a separated form the associated optimal control problem. Full details of the latter can be found in [3].

1. Markov Chain

For $0 \leq i \leq N$ write $e_i = (0, \ldots, 1, \ldots, 0)^*$ for the i-th unit (column) vector in R^{N+1}, and $S = \{e_i,\ 0 \leq i \leq N\}$. We consider a finite state space, continuous time Markov chain $\{X_t\},\ t \geq 0$, defined on a probability space (Ω, F, P); without loss of generality the state space of the chain is taken to be S. Write $p_t^i = P(X_t = e_i)$. We suppose for some family of matrices A_t, that $p_t = (p_t^0, \ldots, p_t^N)^*$ satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = A_t p_t. \tag{1.1}$$

$A_t = (a_{ij}(t)),\ t \geq 0$, is, therefore, the family of Q matrices of the process. We suppose $|a_{ij}(t)| \leq B$ for all i, j and $t \geq 0$. Because A_t is a Q-matrix

$$a_{ii}(t) = -\sum_{i \neq j} a_{ji}(t). \tag{1.2}$$

Write $\Phi(t, s)$ for the fundamental transition matrix associated with A, so, with I the $(N+1) \times (N+1)$ identity matrix

$$\frac{d\Phi}{dt}(t, s) = A_t \Phi(t, s), \qquad \Phi(s, s) = I \tag{1.3}$$

$$\frac{d\Phi}{ds}(t, s) = -\Phi(t, s) A_s, \quad \Phi(t, t) = I.$$

Suppose $\{F_t\}$ is the right continuous, complete filtration generated by X. Then for $0 \le s \le t$, if $X_s = x \in S$,

$$E[X_t \mid F_s] = E_{s,x}[X_t]$$

$$= \Phi(t,s)x.$$

LEMMA 1.1. *$M_t : X_t - X_0 - \int_0^t A_r X_{r-} dr$ is an $\{F_t\}$ martingale.*

Proof. Suppose $0 \le s \le t$. Then

$$E[M_t - M_s \mid F_s] = E\Big[X_t - X_s - \int_s^t A_r X_{r-} dr \mid F_s\Big]$$

$$= E\Big[X_t - X_s - \int_s^t A_r X_r dr \mid X_s\Big],$$

(because $X_r = X_{r-}$ for each ω, except for countably many r),

$$= \Phi(t,s)X_s - X_s - \int_s^t A_r \Phi(r,s) X_s dr$$

$$= 0 \qquad \text{by (1.3).}$$

COROLLARY 1.2. *By variation of constants*

$$X_t = \Phi(t,0)\Big(X_0 + \int_0^t \Phi(r,0)^{-1} dM_r\Big).$$

NOTATION 1.3. *If $x = (x_0, x_1, \ldots, x_N)^* \in R^{N+1}$,* diag *$x$ will be the diagonal matrix with entries from x. For $x, y \in R^{N+1}$ write $x \cdot y = x^* y$ for their scalar (inner) product.*

Consider $0 \le i, j \le N$ and $i \ne j$. Then

$$(X_{s-} \cdot e_i) e_j^* dX_s = (X_{s-} \cdot e_i) e_j^* \Delta X_s$$

$$= (X_{s-} \cdot e_i) e_j^* (X_s - X_{s-})$$

$$= I(X_{s-} = e_i, \ \ X_s = e_j).$$

Define the martingale

$$M_t^{ij} := \int_0^t (X_{s-} \cdot e_i) e_j^* dM_s.$$

(Note the integrand is predictable.) Then

$$M_t^{ij} = \int_0^t (X_{s-} \cdot e_i) e_j^* dX_s - \int_0^t (X_{s-} \cdot e_i) e_j^* A_s X_{s-} ds.$$

Wr'ting $N_t(i,j)$ for the number of jumps of the process X from e_i to e_j up to time t this is

$$\begin{aligned} &= N_t(i,j) - \int_0^t I(X_{s-} = e_i) a_{ji}(s) ds \\ &= N_t(i,j) - \int_0^t I(X_s = e_i) a_{ji}(s) ds \end{aligned}$$

because $X_s = X_{s-}$ for each ω, except for countably many s. Therefore, for $i \neq j$

$$N_t(i,j) = \int_0^t I(X_s = e_i) a_{ji}(s) ds + M_t^{ij}.$$

For a fixed j, $0 \leq j \leq N$, write $N_t(j)$ for the number of jumps into state e_j up to time t. Then

$$N_t(j) = \sum_{\substack{i=1 \\ i \neq j}}^{N} N_t(i,j) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \int_0^t I(X_s = e_i) a_{ji}(s) ds + M_t^j$$

where

$$M_t^j = \sum_{\substack{i=1 \\ i \neq j}}^{N} M_t^{ij}.$$

Finally, write N_t for the total number of jumps (of any kind) of the process X up to time t. Then

$$N_t = \sum_{j=1}^{N} N_t(j) = \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} \int_0^t I(X_s = e_i) a_{ji}(s) ds + Q_t$$

where Q_t is the martingale

$$\sum_{j=1}^{n} M_t^j.$$

However, from (1.2)

$$a_{ii}(s) = -\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ji}(s)$$

so

$$N_t = -\sum_{i=1}^{N} \int_0^t I(X_s = e_i) a_{ii}(s) ds + Q_t. \tag{1.4}$$

2. Filtering

We now consider the recursive estimation of the state X_t given the number of jumps which have occured to time t. Other counting processes, such as $N_t(i,j)$, $N_t(j)$ could be considered as the observation process; for details see [3].

NOTATION 2.1. *Write*

$$h(s, X_s) = -\sum_{i=1}^{N} I(X_s = e_i) a_{ii}(s)$$

and $a(s)$ for the vector $(-a_{00}(s), \ldots, -a_{NN}(s))^*$. Then $h(s, X_s) = a(s) \cdot X_s$. We shall further abbreviate $h(s, X_s)$ as $h(s)$.

We have, therefore, a SIGNAL process

$$X_t = X_0 + \int_0^t A_s X_{s-} ds + M_s \tag{2.1}$$

and an OBSERVATION process

$$N_t = \int_0^t h(s) ds + Q_t. \tag{2.2}$$

Write $\{Y_t\}$ for the right continuous complete filtration generated by N, so $Y_t \subset F_t$ for all t. If $\{\phi_t\}$, $t \geq 0$, is any process write $\hat{\phi}$ for the Y-optional projection of ϕ. Then $\hat{\phi}_t = E[\phi_t \mid Y_t]$ a.s. Similarly, write $\hat{\hat{\phi}}$ for the Y-predictable projection of ϕ. Then $\hat{\hat{\phi}} = E[\phi_t \mid Y_{t-}]$ a.s. From Theorem 6.48 of [2], for almost all ω, $\hat{\phi}_t = \hat{\hat{\phi}}_t$ except for countably many values of t. Therefore

$$\begin{aligned}\int_0^t \hat{\hat{h}}(r, X_r)dr &= \int_0^t \hat{h}(r, X_r)dr \\ &= \int_0^t \hat{h}(r, X_{r-})dr.\end{aligned}$$

Write $\hat{p}_t = \widehat{X}_t = E[X_t \mid Y_t]$ so $\hat{p}_0 = E[X_0] = p_0$ say. Now $h(r) = a(r) \cdot X_r$ so $\hat{h}(r) = a(r) \cdot \hat{p}_r$.

For the vector $h(r)X_r = \text{diag } a(r) \cdot X_r$ we have $\widehat{h(r)X_r} = \text{diag } a(r) \cdot \hat{p}_r$. The innovation process associated with the observations is

$$\begin{aligned}\widetilde{Q}_t &:= N_t - \int_0^t \hat{\hat{h}}(r)dr \\ &= N_t - \int_0^t \hat{h}(r-)dr.\end{aligned}$$

Application of Fubini's theorem shows that $\widetilde{Q}$ is a $\{Y_t\}$ martingale. Therefore,

$$N_t = \int_0^t \hat{h}(r-)dr + \widetilde{Q}_t. \tag{2.3}$$

Similarly, Fubini's theorem shows that the process

$$\widetilde{M}_t := \hat{p}_t - p_0 - \int_0^t A_s \hat{p}_{s-} ds$$

is a square integrable $\{Y_t\}$-martingale. Consequently, $\widetilde{M}$ can be represented as a stochastic integral

$$\widetilde{M}_t = \int_0^t \gamma_r d\widetilde{Q}_r.$$

Therefore,

$$E[X_t \mid Y_t] = \hat{p}_t = p_0 + \int_0^t A_r \hat{p}_{r-} dr + \int_0^t \gamma_r d\widetilde{Q}_r. \tag{2.4}$$

The problem now is to find an explicit from for γ.

THEOREM 2.2.

$$\gamma_r = I(\hat{p}(r-) \cdot a(r) \neq 0)(\hat{p}(r-) \cdot a(r))^{-1}\{\text{diag } a(r) \cdot \hat{p}(r-)$$
$$- (\hat{p}(r-) \cdot a(r))\hat{p}(r-) + A_r \hat{p}(r-)\}.$$

Proof. The product $\hat{p}_t N_t$ is calculated two ways. First consider

$$X_t N_t = \int_0^t X_{r-}(dQ_r + h(r-)dr)$$
$$+ \int_0^t N_{r-}(A_r X_{r-} dr + dM_r) + [X, N]_t.$$

Now X and N jump at the same times, at which $\Delta N = 1$, so

$$[X, N]_t = \sum_{0<r\leq t} \Delta X_r \Delta N_r = \sum_{0<r\leq t} \Delta X_r = X_t - X_0$$
$$= \int_0^t A_r X_{r-} dr + M_t.$$

That is, $\langle X, N\rangle_t = \int_0^t A_r X_{r-} dr$ so

$$X_t N_t = \int_0^t (X_{r-} h(r-) + N_{r-} A_r X_{r-} + A_r X_{r-}) dr + \mu_t \tag{2.5}$$

where μ is an $\{F_t\}$ martingale.

Taking the Y-optional projection of each side of (2.5)

$$\hat{p}_t N_t = \int_0^t (\text{diag } a(r) \cdot \hat{p}_{r-} + N_{r-} A_r \hat{p}_{r-} + A_r \hat{p}_{r-}) dr + H_t^1 \tag{2.6}$$

where H^1 is a square-integrable Y martingale. However, from (2.3) and (2.4)

$$\hat{p}_t N_t = \int_0^t \hat{p}_{r-}\hat{h}(r-)dr + \int_0^t \hat{p}_{r-} d\widetilde{Q}_r + \int_0^t A_r \hat{p}_{r-} N_{r-} dr$$

$$+ \int_0^t \gamma_r N_{r-} d\widetilde{Q}_r + [\hat{p}, N]_t.$$

Now

$$[\hat{p}, N]_t = \sum_{0<r\leq t} \Delta \hat{p}_r \Delta N_r = \sum_{0<r\leq t} \gamma_r dN_r$$

$$= \int_0^t \gamma_r dN_r = \int_0^t \gamma_r d\widetilde{Q}_r + \int_0^t \gamma_r \hat{h}(r-)dr.$$

Therefore,

$$\hat{p}_t N_t = \int_0^t (\hat{p}_{r-}\hat{h}(r-) + A_r \hat{p}_{r-} N_{r-} + \gamma_r \hat{h}(r-))dr + H_t^2, \tag{2.7}$$

where H^2 is a square-integrable Y martingale. The bounded variation process in (2.6) and (2.7) must be equal, so

$$\text{diag } a(r) \cdot \hat{p}_{r-} + N_{r-} A_r \hat{p}_{r-} + A_r \hat{p}_{r-}$$

$$= \hat{p}_{r-}\hat{h}(r-) + A_r \hat{p}_{r-} N_{r-} + \gamma_r \hat{h}(r-).$$

Recalling $\hat{h}(r-) = a(r) \cdot \hat{p}_{r-}$ we have

$$\gamma_r = I_{(p \cdot a \neq 0)}(a(r) \cdot \hat{p}(r-))^{-1}\{\text{diag } a(r) \cdot \hat{p}_{r-}$$

$$- (a(r) \cdot \hat{p}(r-))\hat{p}(r-) + A_r \hat{p}(r-)\}. \tag{2.8}$$

Note for any set $B \in Y_s$

$$E\Big[I_B \int_s^t dN_r\Big] = E\Big[I_B \int_s^t \hat{h}(r-)dr\Big]$$

so γ_r can be taken to be 0 on any set where $\hat{h}(r-) = 0$.

REMARKS 2.3. We have, therefore, that $\hat{p}_t = E[X_t \mid Y_t]$ is given by the equation

$$\hat{p}_t = p_0 + \int_0^t A_r \hat{p}_{r-} dr + \int_0^t \gamma_r (dN_r - a(r) \cdot \hat{p}_{r-} dr)$$

where γ_r is given by (2.8). The disadvantage of this equation is that γ involves the inverse factor $(a(r) \cdot \hat{p}_{r-})^{-1}$.

3. The Zakai Equation

Suppose there is a constant $k > 0$ such that $-a_{ii}(r) > k$ for all i and $r \geq 0$. Then $h(r)^{-1} = (a(r) \cdot X_r)^{-1} < k^{-1}$ for all $r \geq 0$. Define the martingale Λ by

$$\Lambda_t = 1 + \int_0^t \Lambda_{r-}(h(r-)^{-1} - 1) dQ_r \tag{3.1}$$

and introduce a new probability measure P_1 on (Ω, F) by

$$E\Big[\frac{dP_1}{dP} \mid F_t\Big] = \Lambda_t.$$

Then it can be shown that under P_1 the process N_t is a standard Poisson process, and in particular $\overline{Q}_t = N_t - t$ is a martingale. Conversely we can define the (P_1, F) martingale

$$\overline{\Lambda}_t = 1 + \int_0^t \overline{\Lambda}_{r-}(h(r-) - 1) d\overline{Q}_r. \tag{3.2}$$

Then $\Lambda_t \overline{\Lambda}_t = 1$. To obtain the Zakai equation we take P_1 as the reference probability measure and compute expectations under P_1. Write $\Pi(\overline{\Lambda}_t)$ for the Y-optional projection of $\overline{\Lambda}$ under P_1. Then for each $t \geq 0$, $\Pi(\overline{\Lambda}_t) = E_1[\overline{\Lambda}_t \mid Y_t]$ a.s. It can be shown that

$\Pi(\overline{\Lambda}_t) = 1 + \int_0^t \lambda_r d\overline{Q}_r$ where $\lambda_r = \Pi(\overline{\Lambda}_{r-})(\hat{h}(r-) - 1)$. By Baye's rule, for any F_t-measurable random variable ϕ

$$\hat{\phi}_t = E[\phi \mid Y_t] = E_1[\overline{\Lambda}_t \phi \mid Y_t] / E_1[\overline{\Lambda}_t \mid Y_t].$$

Write $\sigma(\phi)_t = E_1[\overline{\Lambda}_t \phi \mid Y_t]$. Then $\sigma(X_t) = E_1[\overline{\Lambda}_t X_t \mid Y_t] = q_t$, say, and $\sigma(1) = \Pi(\overline{\Lambda}_t)$. Now q_t is an unnormalized conditional distribution of X_t given Y_t, because $\hat{p}_t = E[X_t \mid Y_t] = q_t / \Pi(\overline{\Lambda}_t)$. Calculating the product $\Pi(\overline{\Lambda}_t)\hat{p}_t$ we obtain the Zakai equation for q_t:

$$\begin{aligned} q_t = p_0 &+ \int_0^t A_r q_{r-} dr \\ &+ \int_0^t (\text{diag } a(r) - I + A_r) q_{r-} d\overline{Q}_r. \end{aligned} \tag{3.3}$$

This equation is linear in q and the inverse $(a \cdot \hat{p})^{-1}$ has disappeared.

4. Optimal Control

The optimal control of a Markov chain when, say, only the total number of jumps is observed, can be discussed using the Zakai equation (3.3). We see below that this presents the problem in a separated form. We suppose the family of Q-matrix generators $A_t(u)$ depend on a control parameter $u \in U$ (a compact, convex subset of some R^k). Write

$$a(s,u) = (-a_{00}(s,u), \ldots, a_{NN}(s,u))^*$$

and $h(s,u) = a(s,u) \cdot X_s^u$ where the state process X^u is now described by dynamics

$$X_t^u = X_0^u + \int_0^t A_r(u) X_{r-}^u dr + M_t^u$$

for $0 \leq t \leq T$. The set $\underline{U}$ of admissible controls is the set of $\{Y_t\}$-predictable processes with values in U.

Write P_1 for the probability measure under which N_t is a standard Poisson process. Then $\overline{Q}_t = N_t - t$ is a martingale under P_1. For each $u \in \underline{U}$ we define

$$\overline{\Lambda}_t^u = 1 + \int_0^t \overline{\Lambda}_{r-}^u (h(r-, u) - 1) d\overline{Q}_r.$$

$\Pi(\overline{\Lambda}_t^u)$ is the Y-optional projection of $\overline{\Lambda}$ under P_1 and with

$$\sigma(X_t^u) = q_t(u) = E_1[\overline{\Lambda}_t^u X_t^u \mid Y_t]$$

we have as in Section 3 that the unnormalized distribution satisfies:

$$q_t(u) = \Pi(\overline{\Lambda}_t^u)\hat{p}_t(u).$$

Write $B_t(u) = (\operatorname{diag} a(t,u) - I + A_t(u))$. Then for each $u \in \underline{U}$ the unnormalized distribution is given by the Zakai equation

$$q_t(u) = p_0 + \int_0^t A_r(u) q_{r-}(u) du + \int_0^t B_r(u) q_{r-}(u) d\overline{Q}_r.$$

COST: A function on the state space S is represented by a vector

$$\ell = (\ell_0, \ldots, \ell_N)^* \in R^{N+1}.$$

We consider for simplicity just a terminal cost so the control problem is that of choosing $u \in \underline{U}$ so the expected cost

$$J(u) = E[\langle \ell, X_T^u \rangle]$$

is minimized. Now

$$\begin{aligned} J(u) &= E_1[\overline{\Lambda}_T^u \langle \ell, X_T^u \rangle] \\ &= E_1[\langle \ell, \overline{\Lambda}_T^u X_T^u \rangle] \\ &= E_1[\langle \ell, E_1[\overline{\Lambda}_T^u X_T^u \mid Y_t] \rangle] \\ &= E_1[\langle \ell, q_T^u \rangle]. \end{aligned}$$

The control problem has, therefore, been formulated in separated form: find $u \in \underline{U}$ which minimizes

$$J(u) = E_1[\langle \ell, q_T^u \rangle]$$

where for $0 \leq t \leq T$, q_t^u is given by

$$q_t(u) = p_0 + \int_0^t A_r(u) q_{r-}(u) dr + \int_0^t B_r(u) q_{r-}(u) d\overline{Q}_r.$$

Under P_1, $\overline{Q}_t = N_t - t$ is a (P_1, Y) martingale. A novel feature of this partially observed control problem is that there is correlation between the state and observation processes which leads to the presence of the control u in the "diffusion" coefficient B.

A minimum principle, and an equation for the adjoint process, for this problem are derived in [3].

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Control of a Hybrid Conditionally Linear Gaussian Process[1]

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Abstract. A control problem is considered where the coefficients of the linear dynamics are functions of a noisily observed Markov chain. The approximation introduced is to consider these coefficients as functions of the filtered estimate of the state of the chain; this gives rise to a finite dimensional conditional Kalman filter. A minimum principle and a new equation for an adjoint process are obtained.

Key Words. Hybrid control, filtering, minimum principle, adjoint process, separation principle.

1. INTRODUCTION.

The filtering problem, where the state and observation processes are linear equations with Gaussian noise, has as its solution the celebrated result of Kalman. For the related partially observed, linear quadratic control problem the separation principle applies, and the optimal control can be described explicitly as a function of the filtered state estimate.

Suppose, however, the coefficients in the linear dynamics of the state process are functions of a noisily observed Markov chain. Both the filtering problem, and related quadratic control problem, are now nonlinear, and explicit solutions are either difficult to find or of little practical use. The approximation proposed below is to consider the coefficients in the linear dynamics to be functions of the filtered estimate of the Markov chain. In this way a conditional Kalman filter can be written down. These dynamics lead us to consider in Section 3 a conditionally linear, Gaussian control problem. By adapting techniques of Bensoussan, Ref. 1, a minimum principle and a new equation for the adjoint process are obtained.

Other work discussing similar situations and approximations includes the papers, Refs. 2–6 and the recent book, Ref. 7 by Mariton.

2. DYNAMICS.

Consider a system whose state is described by two quantities, a vector $x \in R^d$ and a component σ which can take a finite number of values from a set $S = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$. ($x$ can be thought of as describing the location, velocity etc., of an object; σ might then describe its orientation or some other operating characteristic.)

Let ϕ_i be the function on S described by

$$\phi_i(\sigma) = \begin{cases} 1 & \text{if } \sigma = \sigma_i \\ 0 & \text{if } \sigma \neq \sigma_i \end{cases}$$

and write $\phi(\sigma)$ for the column vector $(\phi_1(\sigma), \ldots, \phi_N(\sigma))'$. ϕ is, therefore, a bijection onto the set of unit column vectors $\{e_1, \ldots, e_N\}$ of R^N, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$.

If σ evolves as a Markov process on S we can, without loss of generality, consider the corresponding process described by ϕ evolving on the set $\{e_1, \ldots, e_N\}$. Write ϕ_t for the state of this process at time t and $p_t = E[\phi_t]$. Suppose the generator of the Markov chain is the Q matrix $Q(t) = (q_{ij}(t)),\ 1 \leq i, j \leq N$, so that p_t satisfies the forward equation

$$\frac{dp_t}{dt} = Q(t)p_t. \tag{1}$$

It follows from (1) that on the family of σ-fields generated by ϕ_t the process M_t is a martingale, where

$$M_t = \phi_t - \phi_0 - \int_0^t Q(s)\phi_s ds. \tag{2}$$

Suppose ϕ is observed only through the noisy process z, where

$$z_t = \int_0^t \Gamma(s, \phi_s)ds + \nu_t. \tag{3}$$

Here ν is a Brownian motion independent of M. Write $\{Z_t\}$ for the right continuous complete family of σ-fields generated by z and $\hat{\phi}_t$ for the Z-optional projection of ϕ, so that

$$\hat{\phi}_t = E[\phi_t \mid Z_t] \quad \text{a.s.}$$

(For a discussion of optional projections see Elliott [4]. Optional projections take care of measurability in both t and w; conditional expectations only concern measurability in w.)

Write $\Delta(s)$ for the vector $(\Gamma(s,e_1),\ldots,\Gamma(s,e_N))$ and diag $\Delta(s)$ for the diagonal matrix with diagonal $\Delta(s)$.

With an innovation process $\hat{\nu}_t$ given by $d\hat{\nu}_t = dz_t - \langle\Delta(t),\hat{\phi}_t\rangle dt$ it is shown in, for example Ref. 8, that the equation for the filtered estimate $\hat{\phi}$ is

$$\hat{\phi}_t = \hat{\phi}_0 + \int_0^t Q(s)\hat{\phi}_s ds + \int_0^t (\text{diag } \Delta(s) - \langle\Delta(s),\hat{\phi}_s\rangle I)\hat{\phi}_s d\hat{\nu}_s. \tag{4}$$

Here $\langle\ ,\ \rangle$ denotes the inner product in R^N and I is the $N\times N$ identity matrix. Equation (4) provides a recursive expression for the best least squares estimate $\hat{\phi}$ of ϕ given the observations z.

Suppose now the x component of the state is described by the equation

$$dx_t = A(\phi_t)x_t dt + \rho_t d\phi_t + B(\phi_t)dw_t. \tag{5}$$

Here $x\in R^d$, $w_t = (w_t^1,\ldots,w_t^n)$ is an n-dimensional Brownian motion independent of M and ν, and $A(\phi_t)$, $B(\phi_t)$ and ρ_t are, respectively, $d\times d$, $d\times n$ and $d\times N$ matrices. Note that

$$A(\phi_t) = \sum_{i=1}^N A(e_i)\langle e_i,\phi_t\rangle$$

$$B(\phi_t) = \sum_{i=1}^N B(e_i)\langle e_i,\phi_t\rangle.$$

Suppose the x process is observed through the observations of y, where

$$dy_t = Hx_t dt + Gd\beta_t. \tag{6}$$

Here $y\in R^p$, $\beta_t = (\beta_t^1,\ldots,\beta_t^m)$ is an m-dimensional Brownian motion independent of M, ν and w and H, (resp. G), is a $p\times d$ (resp. nonsingular $p\times m$) matrix.

Now the y observations also provide information about ϕ, so that altogether we have the states x and ϕ given by (5) and

$$\phi_t = \phi_0 + \int_0^t Q(s)\phi_s ds + M_t, \tag{7}$$

with observations given by (3) and (6). Write $\{\overline{Y}_t\}$ for the right continuous, complete filtration generated by y and z, and denote by a bar the $\overline{Y}$-optional projection of a process so that, for example,

$$\bar{\phi}_t = E[\phi_t \mid \overline{Y}_t] \quad \text{a.s.}$$

Define the H-innovation processes ν^*, β^* by

$$d\nu_t^* = dz_t - \langle \Delta(t), \bar{\phi}_t \rangle dt$$

$$d\beta_t^* = G^{-1}(dy_t - H\bar{x}_t dt).$$

For vectors $x = (x_1, \ldots, x_m)' \in R^m$ and $y = (y_1, \ldots, y_n)' \in R^n$, $x \otimes y$ will denote their tensor product, which can be identified with the $m \times n$ matrix (a_{ij}), $a_{ij} = x_i y_j$. Then the filtered estimate of $\begin{pmatrix} x_t \\ \phi_t \end{pmatrix}$ is given by

$$d\begin{pmatrix} \bar{x}_t \\ \bar{\phi}_t \end{pmatrix} = \begin{pmatrix} \overline{A(\phi_t)x_t} \\ Q_t\bar{\phi}_t \end{pmatrix} dt + \begin{pmatrix} \rho_t Q_t \bar{\phi}_t \\ 0 \end{pmatrix} dt +$$
$$+ \left(\overline{(x_t, \phi_t)' \otimes (Hx_t, \langle \Delta(t), \phi_t \rangle)} - (\bar{x}_t, \bar{\phi}_t)' \otimes (H\bar{x}_t, \langle \Delta(t), \bar{\phi}_t \rangle) \right) \begin{pmatrix} G^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} d\beta_t^* \\ d\nu_t^* \end{pmatrix}.$$

This is a nonlinear equation. However, the approximation we shall make is to suppose that most of our information about ϕ comes from the observations of z and that we can replace ϕ by $\hat{\phi}$ in (5), where $\hat{\phi}$ is given by (4). Note that $\hat{\phi}$ is independent of w and β. We can, therefore, state the following result:

PROPOSITION 2.1. *Suppose the state x_t is approximated by $\tilde{x}_t$ where*

$$d\tilde{x}_t = A(\hat{\phi}_t)\tilde{x}_t dt + \rho_t d\hat{\phi}_t + B(\hat{\phi}_t)dw_t. \tag{8}$$

Here $\hat{\phi}$ is given by (4). Suppose $\tilde{x}$ is observed through the process $\tilde{y}$ where

$$d\tilde{y}_t = H\tilde{x}_t dt + G d\beta_t. \tag{9}$$

Write $\{Y_t\}$ for the right continuous, complete filtration generated by $\tilde{y}$ and $\hat{x}$ for the Y-optional projection of x, so that $\hat{x}_t = E[x_t \mid Y_t]$ a.s. Then

$$d\hat{x}_t = A(\hat{\phi}_t)\hat{x}_t dt + \rho_t d\hat{\phi}_t + P_t H (GG')^{-1} d\hat{\beta}_t \tag{10}$$

$$\hat{x}_0 = E x_0,$$

where

$$G \cdot d\hat{\beta}_t = dy_t - H\hat{x}_t dt \tag{11}$$

and P_t is the matrix solution of the Riccati equation

$$\dot{P}_t = B(\hat{\phi}_t)B(\hat{\phi}_t)' - P_t H'(GG')^{-1} H P_t + A(\hat{\phi}_t)P_t + P_t A(\hat{\phi}_t), \tag{12}$$

$$P_0 = \operatorname{cov} x_0.$$

Proof. Because $\hat{\phi}$ is independent of w and B, $\hat{\phi}_t(w)$ appears as a parameter in (10), so the usual Kalman filter formula applies. Equations (4), (10), (11) and (12) therefore, give a finite dimensional filter for $\hat{x}_t$, which is a conditionally Gaussian random variable given $\hat{\phi}$ and Y_t.

Note that

$$A(\hat{\phi}_t) = \sum_{i=1}^{N} A(e_i)\langle e_i, \hat{\phi}_t\rangle$$

$$B(\hat{\phi}_t) = \sum_{i=1}^{N} B(e_i)\langle e_i, \hat{\phi}_t\rangle.$$

□

REMARKS 2.3.

$$\begin{aligned} d(\bar{x}_t - \hat{x}_t) = {} & (\overline{A(\phi_t)x_t} - A(\hat{\phi}_t)\hat{x}_t)dt \\ & + \rho_t Q_t(\bar{\phi}_t - \hat{\phi}_t)dt + (\overline{x \otimes Hx} - \bar{x} \otimes H\bar{x})G^{-1}d\beta_t^* \\ & + (\overline{x \otimes \langle \Delta(t), \phi_t\rangle} - \bar{x} \otimes \langle \Delta(t), \bar{\phi}_t\rangle)d\nu_t^* \\ & + P_t H(GG')^{-1}d\hat{\beta}_t. \end{aligned}$$

Therefore, with tr denoting the trace of a matrix,

$$\begin{aligned} d(\bar{x}_t - \hat{x}_t)^2 = {} & 2(\bar{x}_t - \hat{x}_t)d(\hat{x}_t - \hat{x}_t) \\ & + tr\big(\overline{x \otimes Hx} - \bar{x} \otimes H\bar{x}\big)(G'G)^{-1}\big(\overline{Hx \otimes x} - H\bar{x} \otimes \bar{x}\big)dt \\ & + tr\big(\overline{\langle \Delta(t), \phi\rangle \otimes x} - \langle \Delta(t), \bar{\phi}\rangle \otimes \bar{x}\big)\big(\overline{x \otimes \langle \Delta(t), \phi\rangle} - \bar{x} \otimes \langle \Delta(t), \bar{\phi}\rangle\big)dt \\ & + trP_t H(GG')^{-2}H'P_t' \cdot dt \\ & + trP_t H(GG')^{-1} \cdot G'(\overline{x \otimes Hx} - \bar{x} \otimes H\bar{x})dt. \end{aligned}$$

Taking expectations the martingale terms disappear and, under integrability or boundedness conditions on the coefficient matrices, an estimate of order $o(t)$ for $E(\bar{x}_t - \hat{x}_t)^2$ can be obtained. However, this does not appear too useful.

3. HYBRID CONTROL.

Suppose the state equation for x now contains a control term, so that

$$dx_t = A(\phi_t)x_t dt + \rho_t d\phi_t + C_t u(t)dt + B(\phi_t)dw_t. \tag{13}$$

The observation process is again y, where

$$dy_t = Hx_t dt + Gd\beta_t. \tag{14}$$

Assume the control parameter u takes values in some space R^k and the admissible control functions are those which are predictable with respect to the right continuous, complete filtration generated by y and $\hat{\phi}$. C_t is a $d \times k$ matrix.

Suppose the control $\{u_t\}$ is to be chosen to minimize the cost

$$J(u) = E\left[\int_0^T (x_t' D_t x_t + u_t' R_t u_t)dt + x_T' F x_T\right]. \tag{15}$$

Here D_t, R_t and F are matrices of appropriate dimensions and R_t is non-singular. Then (7), (3), (13), (14) and (15) describe a nonlinear partially observed stochastic control problem whose solution is in general difficult. To obtain a related completely observed problem the approximation we propose is that ϕ_t is replaced by its filtered estimate $\hat{\phi}_t$ in (13) giving a process $\tilde{x}$, where

$$d\tilde{x}_t = A(\hat{\phi}_t)\tilde{x}_t dt + \rho_t d\hat{\phi}_t + C_t u(t)dt + B(\hat{\varphi}_t)dw_t. \tag{16}$$

The observation process is now $\tilde{y}$, where

$$d\tilde{y}_t = H\tilde{x}_t dt + G \cdot d\beta_t \tag{17}$$

and the admissible controls are the predictable functions with respect to the right continuous, complete filtrations generated by $\tilde{y}$ and z.

The cost is taken to be

$$J(u) = E\Big[\int_0^T (\tilde{x}_t' D_t \tilde{x}_t + u_t' R_t u_t)dt + \tilde{x}_T' F \tilde{x}_T\Big]. \tag{18}$$

Equations (16), (17) and (18) describe a partially observed, linear, quadratic Gaussian control problem which is parametrized by $\hat{\phi}_t$, a process which is independent of w and β. However, we cannot apply the separation principle, as in Ref. 9, because the coefficients in (16) are functions of $\hat{\phi}$. The usual form of the separation principle involves the solution of a Riccati equation solved backwards from the final time T, and we do not know the future values of ϕ. We, therefore, proceed as follows to derive a minimum principle satisfied by an optimal control. We are in effect considering a completely observed optimal control problem with state variables $\hat{\phi}$ and $\hat{x}$, where $\hat{\phi}$ is given by

$$\hat{\phi}_t = \hat{\phi}_0 + \int_0^t Q(s)\hat{\phi}(s)ds + \int_0^t \Pi(s)\hat{\phi}(s)d\hat{\nu}_s \tag{19}$$

and

$$\hat{x}_t = m_0 + \int_0^t A(\hat{\phi}_s)\hat{x}_s ds + \int_0^t \rho_s d\hat{\phi}_s + \int_0^t C_s u_s ds + \int_0^t P_s H(GG')^{-1} d\hat{\beta}_s. \tag{20}$$

Here $\Pi(s) = \text{diag } \Delta(s) - \langle \Delta(s), \hat{\phi}_s \rangle I$ and $m_0 = Ex_0$. Note from (12) that the covariance P_t depends on $\hat{\phi}$ In terms of $\hat{x}$ and P the cost corresponding to control $\{u_t\}$ is, (see Ref. 9),

$$J(u) = E\Big[\int_0^T (\hat{x}_t' D_t \hat{x}_t + u_t' R_t u_t)dt$$
$$+ \hat{x}_T' F \hat{x}_T + \int_0^T tr(P_t D_t)dt + tr(P_T F)\Big].$$

The last two terms do not depend on the control, so we shall consider a problem with dynamics given by (19) and (20), and a cost corresponding to a control u given by

$$J(u) = E\Big[\int_0^T (\hat{x}_t' D_t \hat{x}_t + u_t' R_t u_t)dt + \hat{x}_T' F \hat{x}_T\Big]. \tag{21}$$

Write $\{\widetilde{Y}_t\}$ for the right continuous filtration generated by $\tilde{y}$ and z. Write $L^2_Y[0,T] = \{u(t,\omega) \in L^2([0,T] \times \Omega;\ dt \times dP,\ R^k)$ such that for a.e. t, $u(t,\cdot) \in L^2(\Omega, \widetilde{Y}_t, P, R^k)\}$. Assume U is a compact, convex subset of R^k. Then the set of admissible controls is the set

$$\underline{U} = \{u \in L^2_Y[0,T] : u(t,\omega) \in U \text{ a.e. a.s.}\}.$$

Suppose there is an optimal control u^*. We shall consider perturbations of u^* of the form $u_\theta(t) = u^*(t) + \theta(v(t) - u^*(t))$ where v is any other admissible control and $\theta \in [0,1]$. Then

$$J(u_\theta) \geq J(u^*).$$

Following and simplifying techniques of Bensoussan, Ref. 1, our minimum principle is obtained by investigating the Gateaux derivative of J as a functional on the Hilbert space $L^2_Y[0,T]$. Write $\hat{x}^*$ for the trajectory corresponding to the optimal u^*. Then

$$d\hat{x}_t^* = A(\hat{\phi}_t)\hat{x}_t^* dt + \rho_t d\hat{\phi}_t + C_t u_t^* dt + P_t H (GG')^{-1} d\hat{\beta}_t.$$

Given any sample path $\hat{\phi}$, $\hat{\phi}_t$ will be considered as a time varying parameter. Write $\Phi(\hat{\phi}, t, s)$ for the matrix solution of the equation

$$\frac{d}{dt}\,\Phi(\hat{\phi}, t, s) = A(\hat{\phi}_t)\Phi(\hat{\phi}, t, s)dt$$

with initial condition $\Phi(\phi, s, s) = I$.

LEMMA 3.1. *Suppose $v \in \underline{U}$ is such that $u^*_\theta = u^* + \theta v \in \underline{U}$ for $\theta \in [0, \alpha]$. Write $\hat{x}^\theta$ for the solution of (20) associated with u^*_θ. Then $\psi_t = \left.\frac{\partial \hat{x}^\theta_{0,t}}{\partial \theta}\right|_{\theta=0}$* exists a.s. and

$$\psi_t = \Phi(\hat{\phi}, t, 0) \int_0^t \Phi(\hat{\phi}, s, 0)^{-1} C_s v_s ds. \tag{22}$$

Proof.

$$\hat{x}^\theta_{0,t} = x_0 + \int_0^t A(\hat{\phi}_s)\hat{x}^\theta_s ds + \int_0^t \rho_s d\hat{\phi}_s + \int_0^t C_s(u^*_s + \theta v_s) ds + \int_0^t P_s H(GG')^{-1} d\hat{\beta}_s. \tag{23}$$

From the result of Blagovescenskii and Freidlin, Ref. 10, on the differentiability of solutions of stochastic differential equations with respect to a parameter (23) can be differentiated to give

$$\psi_t = \int_0^t A(\hat{\phi}_s) z_s ds + \int_0^t C_s v_s ds. \tag{24}$$

The solution of (24) is then given by (22). □

NOTATION 3.2. *Consider the martingale*

$$M_t = E\Big[2\int_0^T \hat{x}^{*\prime}_s D_s \Phi(\hat{\phi}, s, 0) ds + 2\hat{x}^{*\prime}_T F \Phi(\hat{\phi}, T, 0) \mid F_t\Big]$$

and write

$$\xi_t = M_t - 2\int_0^t \hat{x}^{*\prime}_s D_s \Phi(\hat{\phi}, s, 0) ds$$

$$p_t = \xi_t \cdot \Phi(\hat{\phi}, s, 0)^{-1} \tag{25}$$

$$\eta_{0,t} = \int_0^t \Phi(\hat{\phi}, s, 0)^{-1} C_s v_s ds.$$

Then there are square integrable processes γ and λ such that the martingale M has a representation as a stochastic integral

$$M_t = E\Big[2\int_0^T \hat{x}^{*\prime}_s D_s \Phi(\hat{\phi}, s, 0) ds + 2\hat{x}^{*\prime}_T F \Phi(\hat{\phi}, T, 0)\Big] + \int_0^t \gamma_s d\hat{\beta}_s + \int_0^t \lambda_s d\hat{\nu}_s.$$

PROPOSITION 3.3.

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = E\Big[\int_0^T (p_s C_s v_s + 2u_s^{*\prime} R_s v_s) ds\Big].$$

Proof.

$$J(u_\theta^*) = E\Big[\int_0^T (\hat{x}_s^{\theta\prime} D_s \hat{x}_s^\theta + u_{\theta s}^{*\prime} R_s u_{\theta s}^*) ds + \hat{x}_T^{\theta\prime} F \hat{x}_T^\theta\Big].$$

Therefore, differentiating we see

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = E\Big[2\int_0^T (\hat{x}_s^{*\prime} D_s z_s + u_s^{*\prime} R_s v_s) ds + 2\hat{x}_T^{*\prime} F z_T\Big]. \tag{26}$$

Using the above notation

$$\psi_t = \Phi(\hat{\phi}, t, 0)\eta_{0,t}$$

$$\xi_T = 2\hat{x}_T^{*\prime} F \Phi(\hat{\phi}, T, 0)$$

so

$$\begin{aligned}\xi_T \eta_{0,T} &= 2\hat{x}_T^{*\prime} F \psi_T \\ &= \int_0^T \xi_s \Phi(\hat{\phi}, s, 0)^{-1} C_s v_s ds - 2\int_0^T \hat{x}_s^{*\prime} D_s \Phi(\hat{\phi}, s, 0)\eta_{0,s} ds \\ &\quad + \int_0^T \gamma_s \eta_{0,s} d\hat{\beta}_s + \int_0^T \lambda_s \eta_{0,s} d\hat{\nu}_s.\end{aligned}$$

Substituting in (26)

$$\begin{aligned}\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} &= E\Big[\xi_T \eta_{0,T} + 2\int_0^T \hat{x}_s^{*\prime} D_s \Phi(\hat{\phi}, s, 0)\eta_{0,s} ds + 2\int_0^T u_s^{*\prime} R_s v_s ds\Big] \\ &= E\Big[\int_0^T \xi_s \Phi(\hat{\phi}, s, 0)^{-1} C_s v_s ds + 2\int_0^T u_s^{*\prime} R_s v_s ds\Big] \\ &= E\Big[\int_0^T (p_s C_s v_s + 2u_s^{*\prime} R_s v_s) ds\Big].\end{aligned}$$

$\square$

Now take v to be of the form $v - u^*$ so that $u_\theta = u^* + \theta(v - u^*) \in \underline{U}$. Applying proposition 3.3 to $J(u_\theta)$ we have the following result.

COROLLARY 3.4. *The optimal control satisfies the minimum principle*

$$p_s C_s u_s^* + 2u_s^{*\prime} R_s u_s^* = \min_{v \in U}(p_s C_s v + 2u_s^{*\prime} R_s v) \text{ a.e. a.s.}$$

Proof. u^* is optimal so $\left.\frac{dJ(u_\theta)}{d\theta}\right|_{\theta=0} \geq 0$, that is for any other admissible control v

$$E\Big[\int_0^T (p_s C_s(u_s^* - v_s) + 2u_s^{*\prime} R_s(u_s^* - v_s))ds\Big] \geq 0.$$

v can equal u^* except on an arbitrary set of the form $A \times [s, s+h]$, $A \in F_s$. Therefore, a.e. dt and a.s. dP,

$$p_s C_s(u_s^* - v_s) + 2u_s^{*\prime} R_s(u_s^* - v_s) \geq 0,$$

where the adjoint variable p is given by (25). $\square$

REMARKS 3.5. From Ref. 11 we know the optimal control u^* is feedback, in the sense that at time t it is a function of the states $\hat{x}_t$ and $\hat{\phi}_t$. However, to avoid derivatives of u^* we suppose u^* always follows the trajectories of $\hat{x}^*$ and $\hat{\phi}$, even if these trajectories are perturbed. By the Markov property we, therefore, have that p_t is a function of $x = \hat{x}_t$ and $\phi = \hat{\phi}_t$. Writing $\Phi = \Phi(\hat{\phi}, t, 0)$ and $y = 2\int_0^t \hat{x}_s^{*\prime} D_s \Phi(\hat{\phi}, s, 0)ds$ we have that $\Psi(t, x, y, \Phi) = p_t(x, \phi) \cdot \Phi + y = M_t$, a martingale.

If we write down the Ito expansion of Ψ the sum of the terms integrated with respect to time must be zero. After division by Φ we have the following equation satisfied by the adjoint process $p = p(t, x, \phi)$.

PROPOSITION 3.6. *Denote the Hessian of p with respect to x (resp. ϕ) by $\frac{\partial^2 p}{\partial x^2}$ (resp. $\frac{\partial^2 p}{\partial \phi^2}$) and write*

$$\Gamma_t = p_t H(GG')^{-1} + \rho_t \Pi(t)\hat{\phi}_t$$

$$\Lambda_t = \Pi(t)\hat{\phi}_t.$$

and $tr(\Gamma_t' \frac{\partial^2 p}{\partial x^2} \Gamma_t)$ *(resp.* $tr(\Lambda_t' \frac{\partial^2 p}{\partial \phi^2} \Lambda_t)$*) for the vector with components* $tr(\Gamma_t' \frac{\partial^2 p_i}{\partial x^2} \Gamma_t)$ *(resp.* $tr(\Lambda_t' \frac{\partial^2 p_i}{\partial \phi^2} \Lambda_t)$*).*

Then

$$\frac{\partial p}{\partial t} + p \cdot A(\hat{\phi}_t) + \frac{\partial p}{\partial x} \cdot (A(\hat{\phi}_t)\hat{x}_t + \rho_t Q\hat{\phi}_t + C_t u_t^*)$$
$$+ \frac{\partial p}{\partial \phi} \cdot Q\hat{\phi}_t + \frac{1}{2} tr(\Gamma_t' \frac{\partial^2 p}{\partial x^2} \Gamma_t) + \frac{1}{2} tr(\Lambda_t' \frac{\partial^2 p}{\partial \phi^2} \Lambda_t) = 0,$$

with terminal condition $p(T, x, \phi) = 2xF$.

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THE ADJOINT PROCESS FOR A
PARTIALLY OBSERVED MARKOV CHAIN

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1. Introduction

A finite state space Markov chain is considered. Without loss of generality its state space can be taken to be the set of unit basis vectors of R^N. On the basis of knowing only the total number of jumps a control problem is discussed in 'separated' form. That is, the Zakai equation for its unnormalized distribution is taken as describing the state of the process. This is a linear, vector equation driven by a standard Poisson process in which the control variable also appears in the 'diffusion' coefficient multiplying the noise term. The controls, similar to those employed by Bismut [2] and Kushner [4], are in the 'stochastic open loop' form. By adapting techniques of Bensoussan [1] and calculating a Gâteaux derivative, the minimum principle satisfied by an optimal control is obtained. Finally, when the optimal control is Markov, the integrand in the martingale representation can be obtained explicitly, and new forward and backward equations satisfied by the adjoint process derived. A full treatment can be found in [3].

2. Dynamics

Without loss of generality, the state space of a finite state Markov chain can be identified with the set $S = \{e_i\}$ of unit vectors $e_i = (0,0,\ldots,1,\ldots,0)^*$ of R^N. Assume, therefore, that X_t, $t \geq 0$, is a Markov process defined on a probability space (Ω, F, P) with state space $S = \{e_1, \ldots, e_N\}$. Write $p_t^i = P(X_t = e_i)$, $1 \leq i \leq N$, and suppose for

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some family of matrices $A_t(u)$ that $p_t = (p_t^1, \ldots, p_t^N)^*$ satisfies the forward Kolmogorov equation

$$\frac{dp_t}{dt} = A_t(u)p_t.$$

Here U, the set of control values, is a compact convex subset of some Euclidean space R^k. Take $0 \leq t \leq T$ and suppose $A_t(u)$ is measurable on $[0,T] \times U$ and continuously 'fferentiable in u.

The set $\underline{U}$ of admissible control functions is the set of U valued functions which depend only on the knowledge of previous jump times. That is, if $T_1, T_2, \ldots$ are the jump times of X and $T_n < t \leq T_{n+1}$, then $u \in \underline{U}$ is a function only if $T_1, \ldots, T_n$ and t. It is easily checked that the process M_t^u is a martingale, where

$$M_t^u := X_t^u - X_0 - \int_0^t A_r(u) X_{r-}^u dr.$$

Write N_t for the total number of jumps to time t. We shall suppose our only knowledge of the Markov chain X comes from observing N. $Y = \{Y_t\}$ is the right continuous, complete filtration generated by N. For $u \in \underline{U}$ set

$$a(s,u) = (-a_{11}(s,u), \ldots, -a_{NN}(s,u))^*$$

and $h(s,u) = a(s,u) \cdot X_s^u$.

The hat $\widehat{\ }$ will denote the Y-optional projection, so that $\hat{p}_s(u) := \widehat{X}_s^u = E[X_s^u \mid Y_s]$ a.s. and $\hat{h}(s,u) = a(s,u) \cdot \hat{p}_s(u)$. Now N can be written

$$N_t = \int_0^t h(s,u)ds + Q_t^u = \int_0^t \hat{h}(s,u)ds + \widetilde{Q}_t^u,$$

where $\widetilde{Q}^u$ is a (Y,P) martingale.

Note that because X^u and N jump at the same time the noises in the state process X^u and observation process N are correlated.

Cost: A real function on the state space $S = \{e_1, \ldots, e_n\}$ is given by a vector $\ell = (\ell_1, \ldots, \ell_N)$. Write $\langle \ell, x \rangle = \ell^* \cdot x$ for the inner product on R^N.

The control problem will be that of choosing $u \in \underline{U}$ so the expected cost $J(u) = E[\langle \ell, X_T^u \rangle]$ is minimized.

Suppose there is a $k > 0$ such that $-a_{ii}(s,u) > k$ for all i, all $s \in [0,T]$ and all $u \in U$. Then $h(s,u)^{-1} = (a(s,u) \cdot X_s^u)^{-1} < k^{-1}$. For $u \in \underline{U}$ consider the martingale Λ^u defined by

$$\Lambda_t^u = 1 + \int_0^t \Lambda_{r-}^u (h(r-,u)^{-1} - 1) dQ_r^u.$$

Then, consider the new measure P_u on (Ω, F) given by $E\left[\frac{dP_u}{dP} \mid F_t\right] = \Lambda_t^u$. It can then be shown that under P_u the process N_t is a standard Poisson process; in particular $\overline{Q}_t = N_t - t$ is a martingale.

The inverse of Λ_t^u is the process $\overline{\Lambda}_t^u$, a (P_u, F) martingale given by the equation

$$\overline{\Lambda}_t^u = 1 + \int_0^t \overline{\Lambda}_{r-}^u (h(s-,u) - 1) d\overline{Q}_r^u.$$

Write $\Pi(\overline{\Lambda}_t^u)$ for the Y-optional projection of $\overline{\Lambda}^u$ under measure P_u and consider the unnormalized conditional expectation of X given by $q_t(u) := E_u[\overline{\Lambda}_t^u X_t \mid Y_t]$. From Bayes' rule $q_t(u) = \Pi(\overline{\Lambda}_t^u)\hat{p}_t(u)$, and with $B_t(u) = (\mathrm{diag}(a(t,u)) + A_t(u) - I)$, $q_t(u)$ is given by the Zakai equation

$$q_t(u) = p_0 + \int_0^t A_r(u) q_{r-}(u) dr + \int_0^t B_r(u) q_{r-}(u) d\overline{Q}_r.$$

Furthermore, the cost can be written

$$J(u) = E_u[\langle \ell, q_T^u \rangle]. \tag{1}$$

The partially observed control problem has, therefore, been formulated in separated form: find $u \in \underline{U}$ which minimizes $J(u)$ given by (1), where q is given by the Zakai equation in which $\overline{Q}_t = N_t - t$ and N_t is a standard Poisson process.

3. Minimum Principle

Suppose, therefore, P_1 is a probability measure under which N is a standard Poisson process, and for $u \in \underline{U}$ consider $q_t(u) \in R^N$ defined by

$$q_t(u) = p_0 + \int_0^t A_r(u) q_{r-}(u) dr + \int_0^t B_r(u) q_{r-}(u) d\overline{Q}_r.$$

The cost corresponding to $u \in \underline{U}$ is $J(u) = E_1[\langle \ell, q_T^u \rangle]$.

For $u \in \underline{U}$ write $\Phi^u(t,s)$ for the solution of the matrix equation

$$d\Phi^u(t,s) = A_t(u)\Phi^u(t-,s)dt + B_t(u)\Phi^u(t-,s)(dN_t - dt)$$

with initial condition $\Phi^u(s,s) = I$.

Furthermore, for $u \in \underline{U}$ consider the matrix $\Psi^u(t,s)$ defined by

$$\Psi^u(t,s) = I - \int_0^t \Psi^u(r-,s)A_r(u)dr - \int_0^t \Psi^u(r-,s)B_r(u)d\overline{Q}_r$$
$$+ \int_0^t \Psi^u(r-,s)B_r^2(u)(I + B_r(u))^{-1}dN_r.$$

Then it is easily checked that

$$\Phi^u(t,s)\Psi^u(t,s) = I.$$

Suppose there is an optimal control $u^* \in \underline{U}$. Write q^* for q^{u^*}, Φ^* for Φ^{u^*}, etc. Consider any other control $v \in \underline{U}$. Then for $\theta \in [0,1]$,

$$u_\theta(t) = u^*(t) + \theta(v(t) - u(t)) \in \underline{U}.$$

Because $U \subset R^k$ is compact, the set $\underline{U}$ of admissible controls can be considered as a subset of the Hilbert space

$$H = L^2[\Omega \times [0,T] : R^k].$$

Now

$$J(u_\theta) \geq J(u^*). \tag{2}$$

Therefore, if the Gâteaux derivative $J'(u^*)$ of J, as a functional on the Hilbert space H, is well defined, differentiating (2) in θ, and evaluating at $\theta = 0$, implies

$$\langle J'(u^*), v(t) - u^*(t) \rangle \geq 0$$

for all $v \in \underline{U}$.

LEMMA 3.1. *Suppose $v \in \underline{U}$ is such that $u^*_\theta = u^* + \theta v \in \underline{U}$ for $\theta \in [0, \alpha]$. Write $q_t(\theta)$ for the solution $q_t(u^*_\theta)$ of the Zakai equation. Then $z_t = \frac{\partial q_t(\theta)}{\partial \theta}\Big|_{\theta=0}$ exists and is the unique solution of the equation*

$$z_t = \int_0^t \Big(\frac{\partial A}{\partial u}(r, u^*)\Big) v_r q^*_{r-} dr + \int_0^t A_r(u^*) z_{r-} dr$$

$$+ \int_0^t \Big(\frac{\partial B}{\partial u}(r, u^*)\Big) v_r q^*_{r-} d\overline{Q}_r + \int_0^t B_r(u^*) z_{r-} d\overline{Q}_r. \tag{3}$$

Proof.

$$q_t(\theta) = p_0 + \int_0^t A_r(u^* + \theta v) q_{r-}(\theta) dr$$

$$+ \int_0^t B_r(u^* + \theta v) q_{r-}(\theta) d\overline{Q}_r.$$

The stochastic integrals are defined pathwise, so differentiating under the integrals gives the result. Comparing (3) and the equation for Φ^u we have the following result by variation of constants.

LEMMA 3.2. *Write*

$$\eta_{0,t} = \int_0^t \Psi^*(r-, 0)\Big(\frac{\partial A}{\partial u}(r, u^*)\Big) v_r q^*_{r-} dr$$

$$+ \int_0^t \Psi^*(r-, 0)\Big(\frac{\partial B}{\partial u}(r, u^*)\Big) v_r q^*_{r-} d\overline{Q}_r$$

$$- \int_0^t \Psi^*(r-, 0)(I + B_r(u^*))^{-1} B_r(u^*)\Big(\frac{\partial B}{\partial u}(r, u^*)\Big) v_r q^*_{r-} dN_r. \tag{4}$$

Then $z_t = \Phi^*(t, 0)\eta_{0,t}$.

Proof. Using the differentiation rule

$$\Phi^*(t, 0)\eta_{0,t} = \int_0^t \Phi^*_- \cdot d\eta + \int_0^t d\Phi^* \eta_- + [\Phi, \eta]_t.$$

Because $\Phi^*_- \Psi^*_- = I$, therefore

$$\Phi^*(t, 0)\eta_{0,t} = \int_0^t \Big(\frac{\partial A}{\partial u}(r, u^*)\Big) v_r q^*_{r-} dr$$

$$+\int_0^t \Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q^*_{r-} d\overline{Q}_r$$

$$-\int_0^t (I+B_r(u^*))^{-1} B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q^*_{r-} dN_r$$

$$+\int_0^t A_r(u)\Phi^*(r-,0)\eta_{0,r-}dr + \int_0^t B_r(u)\Phi^*(r-,0)\eta_{0,r-}d\overline{Q}_r$$

$$+\int_0^t B_r(u)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q^*_{r-} dN_r$$

$$-\int_0^t B_r(u)(I+B_r(u^*))^{-1} B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q^*_{r-} dN_r.$$

Now the dN integrals sum to 0, showing that $\Phi^*\eta$ satisfies the same equation (4) as z. Consequently, by uniqueness, the result follows.

COROLLARY 3.3. $\frac{dJ}{d\theta}(u^*_\theta)\Big|_{\theta=0} = E_1[\langle \ell, \Phi^*(T,0)\eta_{0,T}\rangle]$.

Proof. $J(u^*_\theta) = E_1[\langle \ell, q_t(\theta)\rangle]$. The result follows from lemmas 3.1 and 3.2.

NOTATION 3.4. Write $\Phi^*(T,0)'$ for the transpose of $\Phi^*(T,0)$ and consider the square integrable, vector martingale

$$M_t := E_1[\Phi^*(T,0)'\ell \mid Y_t].$$

Then M_t has a representation as a stochastic integral

$$M_t = E_1[\Phi^*(T,0)'\ell] + \int_0^t \gamma_r d\overline{Q}_r$$

where γ is a predictable R^{N+1} valued process such that

$$\int_0^T E_1[\gamma_r^2]dr < \infty.$$

Under a Markov hypothesis γ will be explicitly determined below.

DEFINITION 3.5. The adjoint process is

$$p_t := \Psi^*(t,0)' M_t.$$

THEOREM 3.6.

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = \int_0^T E_1\Big[\Big\langle p_{r-}, \Big\{\Big(\frac{\partial A}{\partial u}(r,u^*) - (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)\Big\} v_r q_{r-}^*\Big\rangle$$

$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle\Big] dr. \quad (5)$$

Proof. First note that

$$\langle M_T, \eta_{0,T}\rangle = \int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)\Big(\frac{\partial A}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle dr$$

$$+ \int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle d\overline{Q}_r$$

$$- \int_0^T \Big\langle M_{r-}, \Psi^*(r-,0)(I+B_r(u^*))^{-1}$$

$$\times B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle dN_r$$

$$+ \int_0^T \Big\langle \gamma_r, \eta_{0,r-}\Big\rangle d\overline{Q}_r$$

$$+ \int_0^T \Big\langle \gamma_r \Psi^*(r-,0)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle dN_r$$

$$- \int_0^T \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}$$

$$\times B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle dN_r. \quad (6)$$

Taking expectations under P, we have

$$\frac{dJ(u_\theta^*)}{d\theta}\Big|_{\theta=0} = E_1[\langle \ell, \Phi^*(T,0)\eta_{0,T}\rangle]$$
$$= E_1[\langle \Phi^*(T,0)'\ell, \eta_{0,T}\rangle] = E_1[\langle M_T, \eta_{0,T}\rangle].$$

Combining the last two terms in (6) and using the fact that $N_t - t$ is a P_1 martingale, this is

$$= \int_0^T E_1\Big[\Big\langle p_{r-}, \Big(\frac{\partial A}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle$$

$$- \Big\langle p_{r-}, (I+B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle$$

$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I+B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big) v_r q_{r-}^*\Big\rangle\Big] dr.$$

$\square$

Now consider perturbations of u^* of the form $u_\theta(t) = u^*(t) + \theta(v(t) - u(t))$ for $\theta \in [0,1]$ and any $v \in \underline{U}$. Then as noted above

$$\frac{dJ(u_\theta)}{d\theta}\Big|_{\theta=0} = \langle J'(u^*), v(t) - u^*(t)\rangle \geq 0.$$

Expression (5) is, therefore, true when v is replaced by $v - u^*$ for any $v \in \underline{U}$, and we can deduce the following minimum principle.

THEOREM 3.7. *Suppose $u^* \in \underline{U}$ is an optimal control. Then a.s. in w and a.e. in t*

$$\Big\langle p_{r-}, \Big\{\Big(\frac{\partial A}{\partial u}(r,u^*) - (I + B_r(u^*))^{-1}B_r(u^*)\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)\Big\}(v_r - u_r^*)q_{r-}^*\Big\rangle$$

$$+ \Big\langle \gamma_r, \Psi^*(r-,0)(I + B_r(u^*))^{-1}\Big(\frac{\partial B}{\partial u}(r,u^*)\Big)(v_r - u_r^*)q_{r-}^*\Big\rangle \geq 0. \quad (7)$$

4. The Adjoint Process

The process p is the adjoint process. However, (7) also contains the integrand γ. In this section we shall obtain a more explicit expression for γ in the case when u^* is Markov, and also derive forward and backward equations satisfied by p.

ASSUMPTION 4.1. The optimal control u^* is a Markov, feedback control. That is, $u^* : [0,T] \times R^{N+1} \to U$ so that $u^*(s, q_{s-}^*) \in U$.

LEMMA 4.2. *Write δ for the predictable "integrand" such that $\Delta p_t = p_t - p_{t-} = \delta_t \Delta N_t$, i.e., $p_t = p_{t-} + \delta_t \Delta N_t$. Furthermore, write $q_{t-} = q$, $B_{t-}(u^*(t-,q)) = B^*(q_{t-}) = B^*(q)$, and $B_t(u^*(t,q_t)) = B^*(q_t)$. Then*

$$\delta_t(q) = (I + B^*((I + B^*(q))q))^{-1}p_{t-}((I + B^*(q))q) - p_{t-}(q). \quad (8)$$

Proof. Let us examine what happens if there is a jump at time t; that is, suppose $\Delta N_t = 1$. Then $q_t = (I + B^*(q))q$. By the Markov property and from Definition 3.5,

$$p_t = E[D^*(t,T_k)(I + B'_{T_k}(u^*))\dots D^*(T,T_N)\ell \mid Y_t]$$

$$= p_t(q_t) = p_t((I + B^*(q))q)$$

$$= (I + B^*(q_t)')^{-1}p_{t-}((I + B^*(q))q),$$

and the result follows. Heuristically, the integrand δ assumes there is a jump at t; the question of whether there is a jump is determined by the factor ΔN_t.

THEOREM 4.3. *Under Assumption 4.1 and with δ_t given by (8)*

$$\gamma_r = \Phi^*(r-,0)'((I + B_r'(u^*))\delta_r + B_r'(u^*)p_{r-}). \tag{9}$$

Proof. $\Phi^*(t,0)'p_t = M_t = E_1[\Phi^*(T,0)'\ell \mid Y_t] = E_1[\Phi^*(T,0)'\ell] + \int_0^t \gamma_r d\overline{Q}_r$. However, if u^* is Markov the process q^* is Markov, and, writing $q = q_t^*$, $\Phi = \Phi^*(t,0)$,

$$\begin{aligned} E_1[\Phi^*(T,0)'\ell \mid Y_t] &= E_1[\Phi'\Phi^*(T,t)'\ell \mid q, \Phi] \\ &= \Phi' E_1[\Phi^*(T,t)'\ell \mid q]. \end{aligned}$$

Consequently, $p_t = E_1[\Phi^*(T,t)'\ell \mid q]$ is a function only of q, so by the differentiation rule:

$$\begin{aligned} p_t = p_0 &+ \int_0^t \frac{\partial p_{r-}}{\partial q}(Aq_{r-}dr + Bq_{r-}d\overline{Q}_r) + \int_0^t \frac{\partial p_{r-}}{\partial r}\, dr \\ &+ \sum_{0<r\le t} \Big(p_r - p_{r-} - \frac{\partial p_{r-}}{\partial q} Bq_{r-}\Delta N_r\Big) \\ = p_0 &+ \int_0^t \Big[\frac{\partial p_{r-}}{\partial q}(Aq_{r-} - Bq_{r-}) + \delta_r\Big] dr + \int_0^t \delta_r d\overline{Q}_r. \end{aligned}$$

Evaluating the product:

$$\begin{aligned} M_t &= \Phi^*(t,0)'p_t \\ &= p_0 + \int_0^t \Phi^*(r-,0)'\Big[\frac{\partial p_{r-}}{\partial q}(Aq_{r-} - Bq_{r-}) + \delta_r\Big] dr \\ &+ \int_0^t \Phi^*(r-,0)'\frac{\partial p_{r-}}{\partial r}\, dr + \int_0^t \Phi^*(r-,0)'\delta_r d\overline{Q}_r \\ &+ \int_0^t \Phi^*(r-,0)'A'p_{r-}dr + \int_0^t \Phi^*(r-,0)'B'p_{r-}d\overline{Q}_r \\ &+ \int_0^t \Phi^*(r-,0)'B'\delta_r d\overline{Q}_r + \int_0^t \Phi^*(r-,0)'B'\delta_r dr. \end{aligned} \tag{10}$$

However, M_t is a martingale, so the sum of the dr integrals in (10) must be 0, and

$$\gamma_r = \Phi^*(r-,0)'(\delta_r + B_r'(u_r^*)\delta_r + B_r'(u_r^*)p_{r-}).$$

□

THEOREM 4.4. *Suppose the optimal control u^* is Markov. Then* a.s. *in ω* and a.e. *in t, u^* satisfies the minimum principle*

$$\left\langle p_{r-},\ \frac{\partial A}{\partial u}(r,u^*)(v_r-u_r^*)q_{r-}^*\right\rangle+\left\langle \delta_r,\ \frac{\partial B}{\partial u}(r,u^*)(v_r-u_r^*)q_{r-}^*\right\rangle\geq 0. \tag{11}$$

Proof. Substituting γ from (9) into (7), and noting $B(I+B)^{-1}-(I+B)^{-1}B=0$, the result follows. (Substituting for B and δ gives an alternative form.)

We now derive a forward equation satisfied by the adjoint process p:

THEOREM 4.5. *With δ given by (8)*

$$\begin{aligned}p_t=E_1[\Phi^*(T,0)'\ell]&-\int_0^t A_r'(u_r^*)p_{r-}dr\\&-\int_0^t(I+B_r'(u_r^*))\delta_r dr+\int_0^t\delta_r dN_r.\end{aligned} \tag{12}$$

Proof. $p_t=\Psi^*(t,0)'M_t$ and this is

$$\begin{aligned}&=E_1[\Phi^*(T,0)'\ell]-\int_0^t A'\Psi^{*\prime}Mdr\\&-\int_0^t B'\Psi^{*\prime}Md\overline{Q}_r+\int_0^t(I+B')^{-1}B'^2\Psi^{*\prime}MdN_r\\&+\int_0^t\Psi^{*\prime}\gamma_r d\overline{Q}_r-\int_0^t B'\Psi^{*\prime}\gamma_r dN_r\\&+\int_0^t(I+B')^{-1}B'^2\Psi^{*\prime}\gamma_r dN_r\\&=E_1[\Phi^*(T,0)'\ell]-\int_0^t A'p_{r-}dr-\int_0^t B'p_{r-}d\overline{Q}_r\\&+\int_0^t(I+B')^{-1}B'^2p_{r-}dN_r+\int_0^t((I+B')\delta_r+B'p_{r-})d\overline{Q}_r\\&-\int_0^t(I+B')^{-1}B'((I+B')\delta_r+B'p_{r-})dN_r\\&=E_1[\Phi^*(T,0)'\ell]-\int_0^t A'p_{r-}dr\\&+\int_0^t(I+B')\delta_r d\overline{Q}_r-\int_0^t B'\delta_r dN_r.\end{aligned}$$

Therefore, the result follows. □

However, an alternative backward equation for the adjoint process p is obtained from the observation that the sum of the bounded variation terms in (10) must be identically zero. Therefore, we have the following result which appears to be new:

THEOREM 4.6. *With $\mathcal{E}$ given by (8) the Markov adjoint process $p_t(q)$ is given by the backward equation*

$$\frac{\partial p_t}{\partial t} + \frac{\partial p_t}{\partial q} \cdot (A^*(q)' - B^*(q)')q + A^*(q)'p_t + (I + B^*(q)')\delta_t = 0$$

with the terminal condition

$$p_T = \ell.$$

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Time reversal of non-Markov point processes

by

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ABSTRACT. — Time reversal is considered for a standard Poisson process, a point process with Markov intensity and a point process with a predictable intensity. In the latter case an analog of the Fréchet derivative for functionals of a Poisson process is introduced and used in techniques of integration-by-parts to obtain formulate similar to those of Föllmer in the Wiener space situation.

Key words : Point processes, Poisson process, predictable intensity, non-Markov, integration-by-parts, Fréchet derivative.

RÉSUMÉ. — Le retournement du temps est considéré pour un processus de Poisson, un processus ponctuel avec intensité markovienne et un processus ponctuel avec intensité prévisible. Pour le dernier cas, nous introduisons une sorte de dérivée Fréchet pour les fonctionnels d'un processus de Poisson et l'utilisons dans les méthodes d'intégration par parties pour obtenir des formules qui sont similaires à celles de Föllmer pour la situation brownienne.

Classification A.M.S. : 60 G 55, 60 J 75.

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1. INTRODUCTION

The time reversal of stochastic processes has been investigated for some years. One motivation comes from quantum theory, and this is discussed in the book of Nelson [11]. The time reversal of Markov diffusions is treated in, for example, the papers of Elliott and Anderson [4], and Haussman and Pardoux [8]. However, the first discussion of time reversal for a non-Markov process on Wiener space appears in the paper by Föllmer [7], in which he uses an integration-by-parts formula related to the Malliavin calculus.

In the present paper an analog of the Fréchet derivative is introduced for functionals of a Poisson process. The integration-by-parts formula on Poisson space, *see* [6], is formulated in terms of this derivative and counterparts of Föllmer's formulae are obtained.

In Section 2 the time reversed form of the standard Poisson process is derived. Section 3 considers a point (counting) process N with Markov intensity $h(N_t)$, so that $Q_t = N_t - \int_0^t h(N_s)\,ds$ is a martingale, and obtains the reverse time decomposition of Q for $t \in (0, 1]$. Finally, in Section 4, the situation when h is predictable is considered using the "Fréchet" derivative and integration-by-parts techniques mentioned above.

2. TIME REVERSAL UNDER THE ORIGINAL MEASURE

Consider a standard Poisson process $N = \{ N_t : 0 \leqq t \leqq 1 \}$ on $(\Omega, \mathscr{F}, P)$. We take $N_0 = 0$. Let $\{ \mathscr{F}_t \}$ be the right-continuous, complete filtration generated by N. Let $G_t^0 = \sigma \{ N_s : t \leqq s \leqq 1 \}$ and $\{ G_t \}$ be the left-continuous completion of $\{ G_t^0 \}$.

The following result is well known; *see*, for example, Theorem 2.6 in [9]. For completeness we sketch the proof.

THEOREM 2.1. — *Under* P, N *is a reverse time* G_t*-quasimartingale, and it has the decomposition:*

$$N_t = N_1 + M_t - \int_t^1 \frac{N_s}{s}\, ds,$$

where M *is a reverse time* G_t*-martingale.*

Proof. — Since N is Markov, we have, for $\varepsilon > 0$,

$$\begin{aligned} E[N_{t-\varepsilon} - N_t \mid G_t] &= E[N_{t-\varepsilon} - N_t \mid N_t] \\ &= -\frac{\varepsilon}{t} N_t \end{aligned} \tag{2.1}$$

(*see* [5] and [10]). Thus

$$\int_0^t E\,|\,E[N_{s-\varepsilon} - N_s \mid G_s]\,|\, ds = O(\varepsilon).$$

By Stricker's theorem [12], N_t is a reverse time G_t-quasimartingale. Considering approximate Laplacians we see it has the decomposition

$$N_t = N_1 + M_t + \int_t^1 \alpha_s\, ds \tag{2.2}$$

where from (2.1) and (2.2), for almost all t

$$\begin{aligned} \alpha_t &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t E[\alpha_s \mid G_t]\, ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E[N_{t-\varepsilon} - N_t \mid G_t] \\ &= -\frac{N_t}{t}. \quad \square \end{aligned}$$

3. TIME REVERSAL AFTER A CHANGE OF MEASURE: THE MARKOV CASE

Consider a process $h_t = h(N_t)$ which satisfies: There exist positive constants A, $K > 0$ such that $0 < A < h(N_t) \leq K$ for all t, a.s.

Define the family $\{\Lambda_t, 0 \leq t \leq 1\}$ of exponentials:

$$\Lambda_t = \prod_{0 \leq u \leq t} (1 + (h(N_{u-}) - 1)\Delta N_u) \exp\left(\int_0^t (1 - h(N_{u-}))\, du\right).$$

Then Λ is an $(\mathscr{F}_t)$-martingale under P, and is the unique solution of the equation

$$\Lambda_t = 1 + \int_0^t \Lambda_{u-} (h(N_{u-}) - 1)(dN_u - du).$$

Define a new probability measure P^h by

$$\frac{dP^h}{dP} = \Lambda_1.$$

Then under P^h, the process $H_t = N_t - \int_0^t h(N_{u-})\,du$ is an $(\mathscr{F}_t)$-martingale (*see* [3]). Let $\beta(t) = \int_0^t h(N_{u-})\,du$ so that β is positive and increasing in t because h is positive. Write

$$\psi(t) = \beta^{-1}(t),$$
$$N'_t = N_{\psi(t)},$$
$$\mathscr{F}'_t = \mathscr{F}_{\psi(t)}.$$

LEMMA 3.1. – *(N'_t) is a Poisson process under* $(\Omega, \mathscr{F}, (\mathscr{F}'_t), P^h)$.

Proof. – Since $H_t = N_t - \beta(t)$ is an $(\mathscr{F}_t)$-martingale under P^h, $H'_t = H_{\psi(t)} = N_{\psi(t)} - t$ is an $(\mathscr{F}'_t)$-martingale under P^h. By Itô's rule,

$$\begin{aligned} H'^2 &= 2\int_0^t H'_{s-}\,dH'_s + \sum_{s \le t} (\Delta N_{\psi(s)})^2 \\ &= 2\int_0^t H'_{s-}\,dH'_s + N_{\psi(t)}. \end{aligned}$$

Hence $H^2_{\psi(t)} - t$ is also an $(\mathscr{F}'_t)$-martingale under P^h. Therefore, $\{N'_t\}$ is Poisson by Lévy's characterization (Theorem 12.31 in [2]). $\square$

LEMMA 3.2. – N *is Markov under* P^h.

Proof. – Consider any $\varphi \in C_0^\infty(\mathbb{R})$. For $t \ge s$, by Bayes' formula,

$$\begin{aligned} E^h[\varphi(N_t) \mid \mathscr{F}_s] &= \frac{E[\Lambda_t \varphi(N_t) \mid \mathscr{F}_s]}{E[\Lambda_t \mid \mathscr{F}_s]} \\ &= E[\Lambda_s^t \varphi(N_t) \mid \mathscr{F}_s] \\ &= E[\Lambda_s^t \varphi(N_t) \mid N_s], \end{aligned}$$

because N is Markov under P, where

$$\Lambda_s^t = \prod_{s < u \le t} (1 + (h(N_u) - 1)\Delta N_u) \exp\left(\int_s^t (1 - h(N_u))\,du\right).$$

Hence

$$E^h[\varphi(N_t)\,|\,\mathcal{F}_s]=E^h[\varphi(N_t)\,|\,N_s]$$

and N is Markov under P^h. □

Note that

$$H_t=H_1+N_t-N_1+\int_t^1 h(N_s)\,ds. \qquad (3.1)$$

Thus H_t is a reverse time G_t-quasimartingale under P^h if and only if N_t is. To determine the reverse time decomposition we again investigate the approximate Laplacians, as in [4].

THEOREM 3.3.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[N_{t-\varepsilon}-N_t\,|\,G_t]=-E^h\left[h(N_t-1)\frac{N_t}{\displaystyle\int_0^t h(N_u)\,du}\,\middle|\,N_t\right]. \qquad (3.2)$$

Proof. — By Lemma 3.2,

$$E^h[N_t-N_{t-\varepsilon}\,|\,G_t]=E^h[N_t-N_{t-\varepsilon}\,|\,N_t].$$

Consider a bounded, differentiable function φ on $\mathbb{R}$ and its restriction to $\mathbf{Z}$ (the range of N). Now

$$\varphi(N_t)=\varphi(N_{t-\varepsilon})+\int_{t-\varepsilon}^t (\varphi(N_{s-}+1)-\varphi(N_{s-}))\,dN_s.$$

So

$$\begin{aligned}\varphi(N_t)(N_t-N_{t-\varepsilon})&=\int_{t-\varepsilon}^t (N_{s-}-N_{t-\varepsilon})(\varphi(N_{s-}+1)-\varphi(N_{s-}))\,dN_s\\&\quad+\int_{t-\varepsilon}^t \varphi(N_{s-})\,dN_s+\sum_{t-\varepsilon<s\leq t}\Delta\varphi(N_s)\,\Delta N_s\\&=\int_{t-\varepsilon}^t (N_{s-}-N_{t-\varepsilon})(\varphi(N_{s-}+1)-\varphi(N_{s-}))\,dN_s\\&\qquad\qquad+\int_{t-\varepsilon}^t \varphi(N_{s-}+1)\,dN_s.\end{aligned}$$

Since

$$\begin{aligned}H_t&=N_t-\int_0^t h(N_s)\,ds\\&=N_t-\int_0^t h(N_{s-})\,ds\end{aligned}$$

is a martingale under P^h,

$$E^h[\varphi(N_t)(N_t - N_{t-\varepsilon})]$$
$$= E^h\left[\int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\, h(N_{s-})\, ds\right]$$
$$+ E^h\left[\int_{t-\varepsilon}^{t} \varphi(N_{s-}+1)\, h(N_{s-})\, ds\right]. \quad (3.3)$$

Now, if $|\varphi| \leqq C$,

$$\left| E^h\left[\int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\, h(N_{s-})\, ds\right]\right|$$
$$\leqq 2\,KC \int_{t-\varepsilon}^{t} E^h[|N_{s-} - N_{t-\varepsilon}|]\, ds$$
$$\leqq 2\,KC \int_{t-\varepsilon}^{t} E^h\left[\left|N_{s-} - N_{t-\varepsilon} - \int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|\right]$$
$$+ E^h\left[\left|\int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|\right] ds$$
$$\leqq 2\,KC \int_{t-\varepsilon}^{t} \left\{\left[E^h\left|N_{s-} - N_{t-\varepsilon} - \int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|^2\right]^{1/2} + K\,\varepsilon\right\} ds$$
$$\leqq 2\,KC \int_{t-\varepsilon}^{t} \left\{E^h\left[\int_{t-\varepsilon}^{t} h(N_{u-})\, du\right]^{1/2} + K\,\varepsilon\right\} ds$$
$$\leqq 2\,KC \int_{t-\varepsilon}^{t} ((K\,\varepsilon)^{1/2} + K\,\varepsilon)\, ds \leqq K'\,\varepsilon^{3/2} + K''\,\varepsilon^2.$$

Thus from (3.3),

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[\varphi(N_t)(N_t - N_{t-\varepsilon})] = E^h[\varphi(N_{t-}+1)\, h(N_{t-})]$$
$$= E^h[\varphi(N_t+1)\, h(N_t)]. \quad (3.4)$$

However,

$$E^h[\varphi(N_t+1)\, h(N_t)] = E^h[\varphi(N_{\psi(\beta(t))}+1)\, h(N_{\psi(\beta(t))})]$$
$$= E^h[\varphi(N'_{\beta(t)}+1)\, h(N'_{\beta(t)})]$$
$$= E^h[E^h[\varphi(N'_{\beta(t)}+1)\, h(N'_{\beta(t)}) \mid \beta(t)]].$$

And

$$E^h[\varphi(N'_{\beta(t)}+1)\,h(N'_{\beta(t)})\,|\,\beta(t)]$$

$$= \sum_{k=0}^{\infty} \varphi(k+1)\,h(k)\,\frac{\beta(t)^k e^{-\beta(t)}}{k!}$$

$$= \sum_{l=0}^{\infty} \varphi(l)\,h(l-1)\,\frac{\beta(t)^l e^{-\beta(t)}}{l!}\,\frac{l}{\beta(t)}$$

$$= E^h\left[\varphi(N'_{\beta(t)})\,h(N'_{\beta(t)}-1)\,\frac{N'_{\beta(t)}}{\beta(t)}\,\middle|\,\beta(t)\right]$$

$$= E^h\left[\varphi(N_t)\,h(N_t-1)\,\frac{N_t}{\beta(t)}\,\middle|\,\beta(t)\right].$$

Hence,

$$E^h[\varphi(N_t+1)\,h(N_t)] = E^h\left[\varphi(N_t)\,h(N_t-1)\,\frac{N_t}{\int_0^t h(N_u)\,du}\right]. \tag{3.5}$$

Thus from (3.4) and (3.5),

$$\lim_{\varepsilon\downarrow 0} E^h\left[\varphi(N_t)\,\frac{(N_t-N_{t-\varepsilon})}{\varepsilon}\right] = E^h\left[\varphi(N_t)\,h(N_t-1)\,\frac{N_t}{\int_0^t h(N_u)\,du}\right],$$

or

$$\lim_{\varepsilon\downarrow 0} E^h\left[\frac{N_{t-\varepsilon}-N_t}{\varepsilon}\,\middle|\,G_t\right] = -E^h\left[h(N_t-1)\,\frac{N_t}{\int_0^t h(N_u)\,du}\,\middle|\,N_t\right]. \quad \square$$

By Theorem 3.3 and an argument similar to that in [4], we see that N, and hence H, is a reverse time G_t-quasimartingale under P^h, and it has the decomposition

$$H_t = H_1 + M_t + \int_t^1 \alpha_t\, d_t. \tag{3.6}$$

Moreover, we have the following expression for α_t:

THEOREM 3.4. — *The integrand α_t that appears in (3.6) is given by*

$$\alpha_t = h(N_t) - E^h\left[h(N_t-1)\,\frac{N_t}{\int_0^t h(N_u)\,du}\,\middle|\,N_t\right].$$

Proof. – From (3.1) and (3.6),

$$E^h[H_{t-\varepsilon}-H_t\,|\,G_t]=E^h\left[\int_{t-\varepsilon}^{t}\alpha_s\,ds\,|\,G_t\right]$$
$$=E^h[N_{t-\varepsilon}-N_t\,|\,G_t]+E^h\left[\int_{t-\varepsilon}^{t}h(N_s)\,ds\,|\,G_t\right].$$

Thus for almost all t

$$\alpha_t=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E^h\left[\int_{t-\varepsilon}^{t}\alpha_s\,ds\,|\,G_t\right]=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E^h[N_{t-\varepsilon}-N_t\,|\,G_t]+h(N_t).$$

From Theorem 3.3, α_t has the stated form. □

4. TIME REVERSAL AFTER A CHANGE OF MEASURE: THE NON-MARKOV CASE

This section involves an integration by parts for Poisson processes which is effected by using a Girsanov transformation to change the intensity and then compensating by a time change. In contrast, the integration by parts considered in [1] is obtained by introducing a perturbation of the size of the jumps. The topic is further investigated in [6].

Suppose $\{N_t : 0\leqq t\leqq 1\}$ is a Poisson process with jump times $T_1\wedge 1,\ldots,T_n\wedge 1,\ldots$ Let $\{u_t\}$ be a real predictable process satisfying $\{u_t\}$ is positive and bounded a. s.

For $\varepsilon>0$, consider the family of exponentials:

$$\Lambda_t^\varepsilon=\prod_{0\leqq s\leqq t}(1+\varepsilon u_s\Delta N_s)\exp\left(-\int_0^t\varepsilon u_s\,ds\right).$$

Then $\{\Lambda_t^\varepsilon\}$ is an $\{\mathscr{F}_t\}$-martingale with $E[\Lambda_t^\varepsilon]=1$ (*see* [6]). Define a probability measure P^ε on $\mathscr{F}_1$ by

$$\frac{dP^\varepsilon}{dP}=\Lambda_1^\varepsilon.$$

Set

$$\varphi_\varepsilon(t)=\int_0^t(1+\varepsilon u_s)\,ds$$

and write

$$\psi_\varepsilon(t)=\varphi_\varepsilon^{-1}(t)=\int_0^t \frac{1}{1+\varepsilon u_{\psi_\varepsilon(s)}}ds$$

$$\mathscr{F}_t^\varepsilon=\mathscr{F}_{\psi_\varepsilon(t)}.$$

Then the process $N_t^\varepsilon=N_{\psi_\varepsilon(t)}$ is Poisson on $(\Omega, \mathscr{F}, (\mathscr{F}_t^\varepsilon), P^\varepsilon)$ with jump times $\varphi_\varepsilon(T_1)\wedge 1, \ldots, \varphi_\varepsilon(T_n)\wedge 1, \ldots$ (*see* [6]).

For $\{u_t\}$ as above, set $U_t=\int_0^t u_s\,ds$. Suppose $g_s(w)$ is an $\{F_t\}$-predictable function on [0, 1]. Then for $0\leqq s\leqq T_1\wedge 1$,

$$g_s(w)=g(s),$$

and in general, for $T_{n-1}\wedge 1<s\leqq T_n\wedge 1$,

$$g_s(w)=g(s, T_1\wedge 1, \ldots, T_{n-1}\wedge 1).$$

Note that by setting $g_s(0, 0, \ldots)=g(s)$ for $0\leqq s\leqq T_1\wedge 1$, $g_s((s-T_1)\vee 0, \ldots, (s-T_{n-1})\vee 0), 0, 0, \ldots)$ for $T_{n-1}\wedge 1<s\leqq T_n\wedge 1$, etc., such a g can be written in the form

$$g_s(w)=g_s((s-T_1)\vee 0, (s-T_2)\vee 0, \ldots), \qquad s\in[0, 1]. \qquad (4.1)$$

Therefore, we shall consider a predictable function g of this form, and further assume that if

$$g=g_s(t_1, t_2, \ldots),$$

then all the partial derivatives $\frac{\partial g_s}{\partial t_i}$ exist for all s, and there is a constant $K>0$ such that

$$\left|\frac{\partial g_s}{\partial t_i}\right|<K \quad \text{for all } i, \text{and for all } s. \qquad (4.2)$$

We now define the analog of the Fréchet derivative for functionals of the Poisson process.

Write

$$g_s^\varepsilon=g_s((s-\varphi_\varepsilon(T_1))\vee 0, \ldots, (s-\varphi_\varepsilon(T_n))\vee 0, \ldots).$$

Then

$$\left.\frac{\partial g_s^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}=-\sum_{i=1}^\infty \frac{\partial}{\partial t_i}g_s((s-T_1)\vee 0, \ldots, (s-T_n)\vee 0, \ldots) \times\int_0^{T_i} u_r\,dr\, I_{T_i<s}. \qquad (4.3)$$

Define

$$\mu(dt) = -\sum_{i=1}^{\infty} \frac{\partial g_s}{\partial t_i} I_{T_i < s}\, \delta_{T_i}(dt)$$

where δ_{T_i} is the point mass at T_i. Then

$$\begin{aligned} \left.\frac{\partial g_s^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} &= \int_0^s \int_0^t u_r\, dr\, \mu(dt) \\ &= \int_0^s \int_0^s I_{0 \leq r \leq t \leq s}\, u_r\, dr\, \mu(dt) \\ &= \int_0^s \mu([r, s])\, u_r\, dr \\ &= -\int_0^s \sum_{i=1}^{\infty} I_{r \leq T_i < s} \frac{\partial g_s}{\partial t_i} u_r\, dr \\ &= \int_0^s D g_s(\,.\,, [r, s])\, u_r\, dr, \end{aligned}$$

where

$$Dg_s(\,.\,, [r, s]) = -\sum_{i=1}^{\infty} I_{r \leq T_i < s} \frac{\partial g_s}{\partial t_i}.$$

Write

$$Dg_s(\,.\,, U) = \int_0^s Dg_s(\,.\,, [r, s])\, u_r\, dr.$$

Note that

$$Dg_{T_i}(\,.\,, U) = -\sum_{j=1}^{i-1} \frac{\partial g_{T_i}}{\partial t_j} \int_0^{T_j} u_r\, dr. \qquad (4.4)$$

DEFINITION 4.1. – A process $\{g_s\}$ of the form (4.1) is said to be differentiable if it satisfies (4.2) and (4.3) for all u satisfying (i) and (ii) above, and for all s. We call $Dg_s(\,.\,, U)$ the derivative of g_s in the direction U. It is of interest to note that this concept of differentiability of a function of a Poisson process is an analog of the Fréchet derivative of a function of a continuous process. *See* Föllmer [7], where similar formulae arise using the Fréchet derivative.

Now suppose $\{h_s\}$ is a bounded, $\{F_t\}$-predictable process of the form given by (4.1), which satisfies:

(*a*) h is differentiable in the sense of Definition 4.1.

(*b*) $\dfrac{\partial h_s}{\partial s}$ exists, and there exists a constant $A>0$ such that $\left|\dfrac{\partial h_s}{\partial s}\right|<A$ for all s, a.s.

(*c*) There are constants $B>0$, $C>0$ such that $0<B<h_s<C$ for all s, a.s.

It is easy to check that $h_s=h_s((s-T_1)\vee 0, (s-T_2)\vee 0, \ldots)$ is predictable. Consider the family of exponentials:

$$G_t=\prod_{0\leqq s\leqq t}(1+(h_s-1)\Delta N_s)\exp\left(\int_0^t(1-h_s)\,ds\right)$$
$$=\left(\prod_{0\leqq T_i\leqq t}h_{T_i}\right)\exp\left(\int_0^t(1-h_s)\,ds\right). \tag{4.5}$$

Then $\{G_t\}$ is a martingale with $E[G_t]=1$. Since for each fixed ω, if $T_{n-1}(\omega)<t\leqq T_n(\omega)$, G_t is a function of $(t, T_1(\omega), \ldots, T_{n-1}(\omega))$, we see as above that G_t can be considered to be of the form

$$G_t=G_t((t-T_1)\vee 0, \ldots, (t-T_n)\vee 0, \ldots).$$

THEOREM 4.2. — *(G_t) defined in* (4.5) *is differentiable in the sense of Definition* 4.1.

Moreover,

$$DG_1(.,U)G_1^{-1}=\int_0^1\gamma_s u_s G_1^{-1}\,ds$$
$$=\int_0^1\int_r^1\left[\frac{\partial h_s}{\partial s}+\sum_{j=1}^{\infty}I_{\{T_j<s\}}\frac{\partial h_s}{\partial t_j}+Dh_s(.,[r,s])\right]\frac{1}{h_s}dN_s u_r\,dr$$
$$-\int_0^1\int_r^1 Dh_s(.,[r,s])\,ds\,u_r\,dr,\quad \text{a.s.} \tag{4.6}$$

where

$$\gamma_s=-\sum_{i=1}^{\infty}I_{s\leqq T_i\leqq 1}\frac{\partial}{\partial t_i}G_1((1-T_1)\vee 0, \ldots, (1-T_n)\vee 0, \ldots).$$

Proof. — The first identity follows from the definition and properties of the derivative. To determine $DG_t(.,U)$ we calculate the derivative of G_t^ε at $\varepsilon=0$. Write

$$h_s^\varepsilon=h_s((s-\varphi_\varepsilon(T_1))\vee 0, \ldots, (s-\varphi_\varepsilon(T_n))\vee 0, \ldots),$$

so

$$G_t^\varepsilon = \prod_{0 \leqq s \leqq t} (1 + (h_s^\varepsilon - 1)\,\Delta N_{\psi_\varepsilon(s)}) \exp\left(\int_0^t (1 - h_s^\varepsilon)\,ds\right)$$
$$= \Big(\prod_{0 \leqq \varphi_\varepsilon(T_i) \leqq t} h^\varepsilon_{\varphi_\varepsilon(T_i)}\Big) \exp\left(\int_0^t (1 - h_s^\varepsilon)\,ds\right)$$
$$= \Big(\prod_{0 \leqq T_i \leqq \psi_\varepsilon(t)} h^\varepsilon_{\varphi_\varepsilon(T_i)}\Big) \exp\left(\int_0^t (1 - h_s^\varepsilon)\,ds\right).$$

Then

$$\log G_t^\varepsilon = \sum_{i=1}^{\infty} I_{T_i \leqq \psi_\varepsilon(t)} \log h^\varepsilon_{\varphi_\varepsilon(T_i)} + \int_0^t (1 - h_s^\varepsilon)\,ds. \tag{4.7}$$

Differentiate (4.7) with respect to ε, and then set $\varepsilon = 0$, to see

$$DG_t(.,U)\frac{1}{G_t} = \sum_{i=1}^{\infty} \left\{ I_{T_i \leqq t} \left[\frac{\partial h_{T_i}}{\partial t} \int_0^{T_i} u_r\,dr + \sum_{j=1}^{i-1} \frac{\partial h_{T_i}}{\partial t_j} \left(\int_0^{T_i} u_r\,dr - \int_0^{T_j} u_r\,dr \right) \right] \frac{1}{h_{T_i}} \right\} - \int_0^t D\,h_s(.,U)\,ds, \quad \text{a.s.}$$

From (4.4) this is

$$= \sum_{i=1}^{\infty} \left\{ I_{T_i \leqq t} \left[\frac{\partial h_{T_i}}{\partial t} \int_0^{T_i} u_r\,dr + \sum_{j=1}^{i-1} \frac{\partial h_{T_i}}{\partial t_j} \int_0^{T_i} u_r\,dr + D\,h_{T_i}(.,U) \right] \frac{1}{h_{T_i}} \right\}$$
$$- \int_0^t D\,h_s(.,U)\,ds = \int_0^t \left[\frac{\partial h_s}{\partial s} \int_0^s u_r\,dr + \sum_{j=1}^{\infty} I_{\{T_j < s\}} \frac{\partial h_s}{\partial t_j} \int_0^s u_r\,dr + Dh_s(.,U) \right] \frac{1}{h_s}\,dN_s$$
$$- \int_0^t Dh_s(.,U)\,ds. \tag{4.8}$$

(Formally, the differentiation of the indicator functions $I_{T_i \geqq \psi_\varepsilon(t)}$ introduces Dirac measures $\delta(t - T_i)$.) However, $P(T_i = t) = 0$ and we later will take

expectations, so these can be ignored.) From (4.8),

$$\begin{aligned}
\mathrm{DG}_1(.,\mathrm{U})\,\mathrm{G}_1^{-1} &= \int_0^1 \left\{ \frac{\partial h_s}{\partial s} \int_0^s u_r\,dr + \sum_{j=1}^{\infty} \mathrm{I}_{\{\mathrm{T}_j<s\}} \frac{\partial h_s}{\partial t_j} \int_0^s u_r\,dr \right.\\
&\quad \left. + \int_0^s \mathrm{D}h_s(.,[r,s])\,u_r\,dr \right\} \frac{1}{h_s}\,d\mathrm{N}_s - \int_0^1 \int_0^s \mathrm{D}h_s(.,[r,s])\,u_r\,dr\,ds\\
&= \int_0^1 \int_0^1 \mathrm{I}_{0\leq r\leq s\leq 1} \left\{ \frac{\partial h_s}{\partial s} u_r \frac{1}{h_s} + \sum_{j=1}^{\infty} \mathrm{I}_{\{\mathrm{T}_j<s\}} \frac{\partial h_s}{\partial t_j} u_r \frac{1}{h_s} \right.\\
&\quad \left. + \mathrm{D}h_s(.,[r,s])\,u_r \frac{1}{h_s} \right\} dr\,dN_s - \int_0^1 \int_0^1 \mathrm{I}_{0\leq r\leq s\leq 1}\,\mathrm{D}\,h_s(.,[r,s])\,u_r\,dr\,ds\\
&= \int_0^1 \int_r^1 \left[\frac{\partial h_s}{\partial s} + \sum_{j=1}^{\infty} \mathrm{I}_{\{\mathrm{T}_j<s\}} \frac{\partial h_s}{\partial t_j} + \mathrm{D}h_s(.,[r,s]) \right] \frac{1}{h_s}\,d\mathrm{N}_s\,u_r\,dr\\
&\quad - \int_0^1 \int_r^1 \mathrm{D}\,h_s(.,[r,s])\,ds\,u_r\,dr,
\end{aligned}$$

which is (4.6). □

Consider the family of exponentials defined by (4.5) and define a new probability measure P^h on $\mathscr{F}_1$ by:

$$\frac{d\mathrm{P}^h}{d\mathrm{P}} = \mathrm{G}_1.$$

Then (*see* [3]) the process

$$\begin{aligned}
\mathrm{Z}_t &= \mathrm{N}_t - \int_0^t h_s\,ds\\
&= \mathrm{Q}_t - \int_0^t (h_s - 1)\,ds,
\end{aligned} \tag{4.9}$$

where $\mathrm{Q}_t = \mathrm{N}_t - t$, is an $(\mathscr{F}_t)$-martingale under P^h. We want to show that Z_t is a reverse time G_t-quasimartingale under P^h, having the decomposition

$$\mathrm{Z}_t = \mathrm{Z}_1 + \mathrm{M}_t + \int_t^1 \alpha_s\,ds. \tag{4.10}$$

From (4.9), we can write

$$\mathrm{Z}_t = \mathrm{Z}_1 + \mathrm{Q}_t - \mathrm{Q}_1 + \int_t^1 (h_s - 1)\,ds.$$

Now for almost all t

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \mathrm{E}^h \left[\int_{t-\varepsilon}^t (h_s - 1)\,ds \,\middle|\, \mathrm{G}_t \right] = \mathrm{E}^h [h_t - 1 \mid \mathrm{G}_t].$$

Hence, to show that Z_t has the decomposition given by (4.10), it again suffices to consider approximate Laplacien as in [4] and show that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[Q_{t-\varepsilon} - Q_t \mid G_t]$$

exists.

THEOREM 4.3. — *For almost all* $t \in [0, 1]$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[Q_t - Q_{t-\varepsilon} \mid G_t] = \frac{1}{t} E^h[Q_t + a_t \mid G_t] - E^h[b_t \mid G_t] \qquad (4.11)$$

where

$$a_t = \int_0^t \int_s^1 \left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + D h_r(.,[s,r]) \right] \frac{1}{h_r} dN_r \, ds - \int_0^t \int_s^1 D h_r(.,[s,r]) \, dr ds$$

and

$$b_t = \int_t^1 \left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + D h_r(.,[t,r]) \right] \frac{1}{h_r} dN_r - \int_t^1 D h_r(.,[t,r]) \, dr.$$

Proof. — First we note that if $H((1-T_1) \vee 0, \ldots, (1-T_n) \vee 0, \ldots)$ is a square integrable functional and its first partial derivatives are all bounded by a constant, then, using a similar argument as in [6], we have the integration by parts formula

$$E\left[\left(\int_0^1 u_s \, dQ_s\right) H\right] = -E[DH(.,U)] \qquad (4.12)$$

where $DH(.,U)$ is the derivative in direction U of Definition 4.1.

A direct consequence is the product rule

$$E\left[FH\left(\int_0^1 u_s \, dQ_s\right)\right] = -E[FDH(.,U)] - E[HDF(.,U)]. \qquad (4.13)$$

Let $H = G_1$ be the Girsanov density, then (4.13) becomes

$$E^h\left[F \int_0^1 u_s \, dQ_s\right] = -E^h[DF(.,U)] - E^h[FG_1^{-1} DG_1(.,U)]. \qquad (4.14)$$

Now fix $t_0 \in (0, 1)$. Write $T_k(t_0)$ for the k-th jump time of N_t greater than t_0. Suppose F is a bounded and G_{t_0} measurable function. Furthermore, we suppose that F is a differentiable function (in the sense of Definition

4.1) of the form

$$F((1-T_1(t_0))\vee 0,\ldots,(1-T_k(t_0))\vee 0,\ldots),$$

and that the derivatives of F are bounded. Then the measure $DF(.,dt)$ is concentrated on $[t_0, 1]$ and (4.14) holds for such an F. Take $u_s = I_{[t_0-\varepsilon, t_0]}(s)$ in (4.14). For such an F

$$\begin{aligned}DF(.,U) &= \int_{t_0-\varepsilon}^{t_0} DF(.,[r,1])\,dr\\ &= \int_{t_0-\varepsilon}^{t_0} DF(.,[t_0,1])\,dr\\ &= \varepsilon\, DF(.,[t_0,1]).\end{aligned}$$

Therefore, we have from (4.14)

$$E^h[(Q_{t_0}-Q_{t_0-\varepsilon})F] = -\varepsilon E^h[DF(.,[t_0,1])] + E^h\left[FG_1^{-1}\int_{t_0-\varepsilon}^{t_0}\sum_{i=1}^{\infty} I_{s\leq T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\right]. \quad (4.15)$$

From (4.15), for almost all t

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}[(Q_{t_0}-Q_{t_0-\varepsilon})F] = -E^h[DF(.,[t_0,1])] + E^h\left[FG_1^{-1}\sum_{i=1}^{\infty} I_{t_0\leq T_i<1}\frac{\partial G_1}{\partial t_i}\right]. \quad (4.16)$$

Using (4.15) again with $\varepsilon = t_0 = t$, we have

$$-E^h[DF(.,[t,1])] = \frac{1}{t}E^h[Q_t F] - \frac{1}{t}E^h\left[FG_1^{-1}\int_0^t\sum_{i=1}^{\infty} I_{s\leq T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\right]. \quad (4.17)$$

Now let $u_s = I_{[0,t]}(s)$ in Theorem 4.2 to obtain

$$\begin{aligned}-\int_0^t\left(\sum_{i=1}^{\infty} I_{s\leq T_i<1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds &= \int_0^t\int_s^1\left[\frac{\partial h_r}{\partial r}+\sum_{j=1}^{\infty} I_{\{T_j<r\}}\frac{\partial h_r}{\partial t_j}+Dh_r(.,[s,r])\right]\frac{1}{h_r}\,dN_r\,ds\\ &\quad -\int_0^t\int_s^1 Dh_r(.,[s,r])\,dr\,ds.\end{aligned}$$

Hence (4.17) becomes

$$-E^h[DF(.,[t,1])] = \frac{1}{t}E^h[Q_t F] + \frac{1}{t}E^h[a_t F]. \qquad (4.18)$$

Now take $u_s = I_{[t-\varepsilon, t]}(s)$ in Theorem 4.2 to obtain

$$-\int_{t-\varepsilon}^{t}\left(\sum_{i=1}^{\infty} I_{s\leq T_i<1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds$$
$$= \int_{t-\varepsilon}^{t}\int_s^1\left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j<r\}}\frac{\partial h_r}{\partial t_j} + Dh_r(.,[s,r])\right]\frac{1}{h_r}dN_r\,ds$$
$$-\int_{t-\varepsilon}^{t}\int_s^1 Dh_r(.,[s,r])\,dr\,ds. \qquad (4.19)$$

Multiply both sides of (4.19) by F, and then take expectations

$$-E^h\left[F\int_{t-\varepsilon}^{t}\left(\sum_{i=1}^{\infty} I_{s\leq T_i<1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds\right] = E^h\left[F\int_{t-\varepsilon}^{t} b_s\,ds\right]. \qquad (4.20)$$

Divide both sides of (4.20) by ε, and then let 0, to obtain for almost all t

$$-E^h\left[F\left(\sum_{i=1}^{\infty} I_{t\leq T_i<1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\right] = E^h[b_t F]. \qquad (4.21)$$

Combining (4.16), (4.18) and (4.21), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E^h[(Q_t - Q_{t-\varepsilon})F] = \frac{1}{t}E^h[(a_t + Q_t)F] - E^h[b_t F].$$

Thus we have proved (4.11). □

As a consequence of Theorem 4.3, Z_t is a reverse time G_t-quasimartingale having the decomposition given by (4.10). It follows immediately that the integrand α_t in (4.10) is given by

$$\alpha_t = E^h[b_t + h_t - 1 \mid G_t] - \frac{1}{t}E^h[a_t + Q_t \mid G_t].$$

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Orthogonal Martingale Representation

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Abstract

Stochastic integrals with respect to a martingale X often involve a predictable process integrated against the continuous martingale component X^c together with terms which are integrals of the compensated random measures associated with the jumps. The latter are related to 'optional' stochastic integrals. The main result of this paper relates such a stochastic integral with the sum of a predictable stochastic integral of X and an orthogonal martingale. The result has applications in the hedging of contingent claims in finance.

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1 Introduction

For a real local martingale

$$X_t = X_0 + M_t^c + M_t^d$$

write $\mu = \mu^X$ for the random measure associated with the jumps of X (see Jacod [7]), and $\nu = \mu^P$ for its predictable compensator.

Consider a local martingale of the form

$$N_t = N_0 + \int_0^t \phi_s dM_s^c + \int_0^t \int_R \psi_s(y)(\mu(dy,ds) - \nu(dy,ds)) \qquad (1)$$

where ϕ and ψ are suitable integrands. Projecting N on X we can write

$$N_t = N_0 + \int_0^t \gamma_s dX_s + \Gamma_t \qquad (2)$$

where Γ is a local martingale orthogonal to X, in the sense that the product ΓX is a local martingale. In this paper our purpose is to determine explicit formulae for γ in terms of ϕ and ψ, both in the general case and for Markov diffusions with jumps.

These results have applications in the hedging of contingent claims in finance. See [5] and [6]. For example, suppose that the martingale X_t represents the price of some asset at time t and that for $T > 0$ a contingent claim is given by $H(X_T)$, where H is a function such that $H(X_T)$ is a real, square integrable random variable. Suppose at time t we invest amount ξ_t in the asset and an amount η_t in a riskless bond with zero interest rate and price $Y = 1$. Then the value of our portfolio at time t is

$$V_t = \xi_t X_t + \eta_t Y = \xi_t X_t + \eta_t.$$

We assume ξ is predictable with respect to the filtration $\{F_t\}$ generated by X, and η is adapted. The accumulated gain from the asset price fluctuations up to time t is the stochastic integral $\int_0^t \xi_s dX_s$. Then the cost accumulated to time t by using the investment strategy (ξ_t, η_t) is

$$C_t = V_t - \int_0^t \xi_s dX_s, \qquad 0 \le t \le T.$$

We want our investment strategy to duplicate the contingent claim, so for a strategy (ξ, η) to be admissible we also require

$$V_T = \xi_T X_T + \eta_T = H(X_T).$$

Now suppose $H(X_T)$ can be represented as a (predictable) stochastic integral

$$H(X_T) = E[H(X_T)] + \int_0^T \xi_s^H dX_s \quad \text{a.s.} \tag{3}$$

for some (predictable) integrand ξ^H. Then let us take an investment strategy (ξ, η) and value process V defined by:

$$\xi_t = \xi_t^H, \qquad \eta_t = V_t - \xi_t X_t$$

and

$$V_t = E[H(X_T)] + \int_0^t \xi_s^H dX_s.$$

We have $V_T = H(X_T)$, so the strategy is admissible and for all $t \in [0, T]$

$$C_t = C_T = C_0 = E[H(X_T)].$$

That is the strategy is self-financing because, apart from the initial cost $C_0 = E[H(X_T)]$, no additional costs arise and no risks are involved.

Conversely, if there is a self-financing strategy (ξ, η),

$$V_t = C_t + \int_0^t \xi_s dX_s = C_0 + \int_0^t \xi_s dX_s,$$

so V_t is a martingale. Therefore,

$$V_t = E[V_T \mid F_t] = E[H(X_T) \mid F_t]$$

and

$$V_0 = C_0 = E[H(X_T)],$$

so the martingale V has the representation

$$V_t = E[H(X_T)] + \int_0^t \xi_s dX_s. \tag{4}$$

The existence of a self-financing strategy is, therefore, equivalent to the representation of the martingale $E[H(X_T) \mid F_t]$ in the form (4) for some predictable integrand ξ. In general, a representation in this form is not available. In this case we can proceed as follows (cf. [5] and [6]).

Definition 1.1 *An admissible investment strategy (ξ, η) is said to be mean self-financing if the corresponding cost process C is a martingale. That is, for $t \leq T$,*

$$E[C_T - C_t \mid F_t] = 0 \quad \text{a.s.}$$

In this case, by definition

$$V_t = C_t + \int_0^t \xi_s dX_s = E[C_T \mid F_t] + \int_0^t \xi_s dX_s.$$

Therefore, V is a martingale for an admissible mean self-financing strategy. Consequently,

$$V_0 = C_0 = E[H(X_T)]$$

and

$$V_t = E[H(X_T) \mid F_t] = E[H(X_T)] + K_t + \int_0^t \xi_s dX_s$$

where K_t is the martingale $E[C_T \mid F_t] - C_0$. Note that, if (ξ, η) is an admissible, mean self-financing strategy, $V_t = E[H(X_T) \mid F_t]$ is independent of ξ. However,

$$C_t = C_t^\xi = V_t - \int_0^t \xi_s dX_s$$

does depend on ξ, as does K above. Therefore, $K_t^\xi = E[C_T^\xi \mid F_t] - C_0^\xi$ and each admissible, mean self-financing strategy (ξ, η) gives rise to a decomposition:

$$V_t = E[H(X_T) \mid F_t] = E[H(X_T)] + K_t^\xi + \int_0^t \xi_s dX_s. \tag{5}$$

Definition 1.2 *For each admissible mean self-financing strategy the remaining risk is defined to be*

$$R_t^\xi = E[(C_T^\xi - C_t^\xi)^2 \mid F_t].$$

Consider the unique Kunita–Watanabe decomposition

$$V_t = E[H(X_T)] + \Gamma_t + \int_0^t \xi_s^* dX_s$$

where Γ is a martingale orthogonal to X and ξ^* is a predictable integrand. Now define an investment strategy (ξ^*, η^*) by putting

$$V_t = E[H(X_T)] + \Gamma_t + \int_0^t \xi_s^* dX_s \quad \text{and} \quad \eta_t^* = V_t - \xi_t^* X_t. \tag{6}$$

Then (ξ^*, η^*) is an admissible, mean self-financing strategy which minimizes the remaining risk R_t. To see this note that for any other admissible, mean self-financing strategy (ξ, η):

$$C_T^\xi - C_t^\xi = K_T^\xi - K_t^\xi.$$

However, from (5) and (6)

$$K_T^\xi - K_t^\xi + \int_t^T \xi_s dX_s = \Gamma_T - \Gamma_t + \int_t^T \xi_s^* dX_s.$$

Therefore, because Γ is orthogonal to X,

$$E[(C_T^\xi - C_t^\xi)^2 \mid F_t] = E[(\Gamma_T - \Gamma_t)^2 \mid F_t] + E\Big[\int_0^T (\xi_s^* - \xi_s)^2 d\langle X, X\rangle_s \mid F_t\Big]$$

and this is minimized when $\xi = \xi^*$. Consequently, the unique admissible, risk minimizing investment strategy is (ξ^*, η^*), where ξ^* is the predictable integrand arising in the representation (6).

This discussion indicates why decompositions such as (6), together with an explicit formula for the integrand, are of interest in finance. Representations such as (1) arise when the asset price $X_t = X_0 + M_t^c + M_t^d$ also involves random disturbances of jump type. In that case, a contingent claim typically admits a representation as in (1)

$$H(X_T) = E[H(X_T)] + \int_0^T \phi_s dM_s^c + \int_0^T \int_R \psi_s(y)(\mu(dy, ds) - \nu(dy, ds)). \tag{7}$$

Then ϕ_s and $\psi_s(y)$ for each $y \in R$, (or, at least, for each y in the support of $\mu = \mu^X$), must represent amounts invested in different assets in order to duplicate (i.e., represent) the claim $H(X_T)$. However, if the only assets available are X and $Y = 1$, we must consider the alternative representation (2)

$$H(X_T) = E[H(X_T)] + \int_0^T \gamma_s dX_s + \Gamma_t.$$

Then γ will generate the risk minimizing mean self-financing investment strategy described above.

In particular, even though (after a Girsanov change of measure) the Markov diffusion process considered by Aase in [1] is complete in a mathematical sense, that is, contingent claims have a representation of the form (7), it is not complete in the financial sense; that is, they do not necessarily admit a decomposition (3). To replicate the claims in Aase's model an uncountable number of additional artificial assets would be required. Clearly this is not realistic.

Orthogonal martingale representation after a Girsanov change of measure will be discussed in another paper.

2 Orthogonal Projection

Consider a real local martingale

$$X_t = X_0 + M_t^c + M_t^d.$$

Suppose

$$\mu = \mu^X(dy, dt) = \sum_{s>0} I_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(dy, dt)$$

and

$$\nu = \nu(dy, dt) = \mu^P.$$

Write $\{F_t\}$ for the right continuous, complete filtration generated by X. Consider a process N which is a stochastic integral of the form

$$N_t = N_0 + \int_0^t \phi_s dM_s^c + \int_0^t \int_R \psi_s(y)(\mu(dy, ds) - \nu(dy, ds)) \tag{8}$$

for suitable integrands ϕ and ψ. What we wish to do is write

$$N_t = N_0 + \int_0^t \gamma_s dX_s + \Gamma_t \tag{9}$$

where γ is a predictable integrand and Γ is a local martingale orthogonal to X. From Jacod [7] we know that the stochastic integral $\int\int \psi(d\mu - d\nu)$ in (8) is related to an optional stochastic integral with respect to $M^d = \int_0^t \int_R y(\mu(dy, ds) - \nu(dy, ds))$; consequently we are relating the optional integrals in (8) to the predictable integral in (9).

Proposition 2.1 *Assume there is a reference measure $v = v(w, ds)$ such that $\langle M^c, M^c\rangle$ and λ are absolutely continuous with respect to v, where*

$$\nu(dy, ds) = m(s, dy)\lambda(ds).$$

Write $\rho_s = d\langle M^c, M^c\rangle/dv$ and $\lambda_s = d\lambda/dv$. Then if N_t is the martingale given by (8), the process γ in (9) is

$$\gamma_s = \frac{\rho_s\phi_s + \lambda_s \int_R y\psi_s(y)m(s, dy)}{\rho_s + \lambda_s \int_R y^2 m(s, dy)}. \tag{10}$$

Proof. Note if γ is the predictable integrand of (9)

$$\begin{aligned} \Lambda_t &= \int_0^t \gamma_s dX_s = \int_0^t \gamma_s dM_s^c + \int_0^t \gamma_s dM_s^d \qquad (11) \\ &= \int_0^t \gamma_s dM_s^c + \int_0^t \int_R \gamma_s y(\mu(dy, ds) - \nu(dy, ds)). \end{aligned}$$

From (8) and (9)

$$\Gamma_t = \int_0^t (\phi_s - \gamma_s) dM_s^c + \int_0^t \int_R (\psi_s(y) - \gamma_s y)(\mu(dy, ds) - \nu(dy, ds)).$$

The martingales Γ and X are orthogonal if $[\Gamma, X]$ is a martingale (see Dellacherie and Meyer VIII.41, [3]). However, writing X as

$$X_t = X_0 + M_t^c + \int_0^t \int_R y(\mu(dy, ds) - \nu(dy, ds))$$

we have

$$
\begin{aligned}
[\Gamma, X]_t &= \int_0^t (\phi_s - \gamma_s) d\langle M^c, M^c\rangle_s + \int_0^t \int_R y(\psi_s(y) - \gamma_s y)\mu(dy, ds) \\
&= \int_0^t (\phi_s - \gamma_s)\rho_s ds + \int_0^t \int_R y(\psi_s(y) - \gamma_s y)(\mu(dy, ds) - \nu(dy, ds)) \\
&\quad + \int_0^t \int_R y(\psi_s(y) - \gamma_s y) m(s, dy)\lambda_s ds.
\end{aligned}
$$

This is a martingale if and only if

$$
\int_0^t (\phi_s - \gamma_s)\rho_s ds + \int_0^t \int_R y(\psi_s(y) - \gamma_s y) m(s, dy)\lambda_s ds
$$

is the null process, which is the case if and only if the integrand is zero. Therefore,

$$
(\phi_s - \gamma_s)\rho_s + \lambda_s \int_R y(\psi_s(y) - \gamma_s y) m(s, dy) = 0
$$

and (10) follows.

Remarks 2.2. 1) If $M^d = 0$, so that $\mu = \nu = 0$, then $\gamma = \phi$.

2) Note

$$
\begin{aligned}
\int_R y\psi_s(y) m(s, dy) &= E[\Delta X_s \Delta N_s \mid F_{s-}] \\
&= {}^p(\Delta X \Delta N)_s
\end{aligned}
$$

and

$$
\int_R y^2 m(s, dy) = E[\Delta X_s^2 \mid F_{s-}] = {}^p(\Delta X^2)_s,
$$

where ${}^p(\ \ )$ denotes the predictable projection.

3) If $M^c = 0$,

$$
\gamma_s = \frac{\int_R y\psi_s(y) m(s, dy)}{\int_R y^2 m(s, dy)} = \frac{E[\Delta X_s \Delta N_s \mid F_{s-}]}{E[\Delta X_s^2 \mid F_{s-}]} = \frac{{}^p(\Delta X \Delta N)_s}{{}^p(\Delta X^2)_s}
$$

so γ can be interpreted as the regression of the jumps of N on the jumps of X.

4) With appropriate interpretation of products as tensor products in R^m, the same expression for γ is valid when X is an m-dimensional martingale.

3 Representation Results

Consider a real, local martingale

$$X_t = X_0 + M_t^c + M_t^d$$

and let $\mu = \mu^X$, $\nu = \mu^p$. Suppose $F \in C^{1,2}$, the space of functions continuously differentiable in t and twice continuously differentiable in x. Then the differentiation rule gives (see Jacod [7])

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{\partial F}{\partial s}(s,X_{s-})ds + \int_0^t \frac{\partial F}{\partial x}(s,X_{s-})dM_s^c \\ &+ \int_0^t \int_R (F(s,X_{s-}+y) - F(s,X_{s-}))(\mu(dy,ds) - \nu(dy,ds)) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_{s-})d\langle M^c, M^c\rangle_s \\ &+ \int_0^t \int_R \Big(F(s,X_{s-}+y) - F(s,X_{s-}) - \frac{\partial F}{\partial x}(s,X_{s-})\Big)\nu(dy,ds). \end{aligned} \tag{12}$$

Suppose X is Markov. For a time $T > 0$ and an integrable C^2 function $H(\cdot)$, consider the random variable $H(X_T)$ and the martingale

$$N_t = E[H(X_t) \mid F_t].$$

Because X is Markov

$$N_t = E[H(X_T) \mid X_t] = V(t, X_t), \quad \text{say.}$$

The following representation result appears to be part of the folklore.

Proposition 3.1 *Suppose V is $C^{1,2}$, that is continuously differentiable in t and twice continuously differentiable in x. Then the martingale N is given by the stochastic integral representation*

$$\begin{aligned} N_t = E[H(X_T)] &+ \int_0^t \frac{\partial V}{\partial x}(s,X_{s-})dM_s^c \\ &+ \int_0^t \int_R (V(s,X_{s-}+y) - V(s,X_{s-})) \\ &\times(\mu(dy,ds) - \nu(dy,ds)). \end{aligned} \tag{13}$$

Furthermore, suppose

$$\langle M^c, M^c\rangle_t = \int_0^t \rho_s ds$$

and

$$\nu(dy, ds) = m(s, dy)\lambda_s ds.$$

Then V is the solution of the backward Kolmogorov equation

$$\begin{aligned}&\frac{\partial V}{\partial s} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\rho_s \\ &+\int_R \Big(V(s, X_{s-} + y) - V(s, X_{s-}) - \frac{\partial V}{\partial x}(s, X_{s-})y\Big)m(s, dy)\lambda_s \\ &= 0 \end{aligned} \tag{14}$$

with terminal condition

$$V(T, X_T) = H(X_T).$$

Proof. The result follows by expanding $V(t, X_t)$ by the Ito rule (12) and observing that, because $V(t, X_t) = N_t$ is a martingale, the sum of the bounded variation terms must be the null process.

Remarks 3.2. Often, (see Example 3.4), the differentiability of V follows from flow properties. In the pure jump case only increments of V enter in (13).

Corollary 3.3 *Write* $\Delta V_s(y) = V(s, X_{s-} + y) - V(s, X_{s-})$.

Then from Proposition 2.1 N_t can be written

$$N_t = E[H(X_T)] + \int_0^t \gamma_s dX_s + \Gamma_t$$

where

$$\gamma_s = \frac{\rho_s \frac{\partial V}{\partial x} + \lambda_s \int_R y\Delta V_s(y) m(s, dy)}{\rho_s + \lambda_s \int_R y^2 m(s, dy)} \tag{15}$$

and Γ is a martingale orthogonal to X.

Example 3.4. Suppose X is a Markov diffusion:

$$\begin{aligned} X_t = X_0 \quad &+ \quad \int_0^t g(s, X_{s-})dB_s \\ &+ \quad \int_0^t \int_R h(s, X_{s-}, y)(\tilde{\mu}(dy, ds) - \tilde{\nu}(dy, ds)). \quad (16) \end{aligned}$$

Here $\tilde{\mu}$ is a random measure and $\tilde{\nu} = \tilde{\mu}^p$; note $\tilde{\mu}$ is not μ^X. Suppose $\tilde{\nu}(dy, ds) = m(s, X_{s-}, dy)\lambda_s(X_{s-})ds$. Write $\xi_{r,t}(x)$ for the solution of (16) starting at time r in position x, so that

$$\begin{aligned} \xi_{r,t}(x) = x \quad &+ \quad \int_r^t g(s, \xi_{r,s-}(x))dB_s \quad (17) \\ &+ \quad \int_r^t \int_R h(s, \xi_{r,s-}(x), y)(\tilde{\mu}(dy, ds) - \tilde{\nu}(dy, ds)). \end{aligned}$$

Suppose g, h, m and λ and their first two derivatives in x are measurable with linear growth in the x variable. Then from the theory of stochastic flows, see [2], it is known there is a set $A \subset \Omega$ of measure zero such that the map $(r, t, x) \to \xi_{r,t}(x)$ is twice differentiable in x with derivative $\dfrac{\partial \xi_{r,t}(x)}{\partial x} = D_{r,t}$. Again write $\{F_t\}$ for the right continuous σ-field generated by X and suppose H is a C^2, integrable function. For $T > 0$ consider the right continuous martingale

$$\begin{aligned} N_t &= E[H(\xi_{0,T}(x_0)) \mid F_t] \\ &= E[H(\xi_{t,T}(X_t)) \mid X_t] = V(t, X_t). \end{aligned}$$

From the differentiability of the flow

$$\frac{\partial V}{\partial x}(t, X_t) = E\Big[\frac{\partial H}{\partial x}(X_T)D_{t,T} \mid F_t\Big].$$

Substituting in (13) and (15) we have the representations

$$\begin{aligned} N_t \quad = \quad & E[H(X_T)] + \int_0^t E\Big[\frac{\partial H}{\partial x}(X_T)D_{0,T} \mid F_{s-}\Big] D_{0,s-}^{-1} g(s, X_{s-})dB_s \\ & + \int_0^t \int_R (E[H(\xi_{s-,T}(X_{s-} + y)) - H(\xi_{s-,T}(X_{s-})) \mid X_{s-}] \\ & \times h(s, \xi_{0,s-}(x_0), y)(\tilde{\mu}(dy, ds) - \tilde{\nu}(dy, ds)) \end{aligned}$$

and from Corollary 3.3 this is

$$= E[H(X_T)] + \int_0^t \gamma_s dX_s + \Gamma_t$$

where

$$\gamma_s = \Big\{ g(s, X_{s-})^2 E\Big[\frac{\partial H}{\partial x}(X_T) D_{0,T} \mid F_{s-}\Big] D_{0,s}^{-1}$$
$$+ \lambda_s(X_{s-}) \int_R y E[H(\xi_{s-,T}(X_{s-} + y)) - H(\xi_{s-,T}(X_{s-})) \mid X_{s-}\Big]$$
$$\times h(s, \xi_{0,s-}(x_0), y) m(s, \xi_{0,s-}(x_0), dy)\Big\}$$
$$\times \Big[g(s, X_{s-})^2 + \lambda_s(X_{s-})$$
$$\times \int_R y^2 h(s, \xi_{0,s-}(x_0), y)^2 m(s, \xi_{0,s-}(x_0), dy)\Big]^{-1}.$$

Example 3.5. The random measure in (16) could be a Poisson random measure. However, for simplicity suppose it is a finite sum of independent Poisson processes N_t^i, $i = 1, \dots, n$ with time varying jump sizes a_t^i and intensities λ_s^i. Suppose $g(s, X_{s-}) = \sigma X_{s-}$ and $h(s, X_{s-}, y) = X_{s-}$, so that X_t, (representing an asset price under a 'risk neutral' measure), is given by the following "log Poisson plus log normal" equation

$$X_t = X_0 + \sigma \int_0^t X_{s-} dB_s + \sum_{i=1}^n \int_0^t X_{s-} a_s^i (dN_s^i - \lambda_s^i ds). \tag{18}$$

Suppose for an integrable C^2 function H, $H(X_T)$ represents a contingent claim depending on the asset price at time $T > 0$. Then

$$\begin{aligned} N_t &= E[H(X_T) \mid F_t] \\ &= E[H(X_T)] + \int_0^t \gamma_s dX_s + \Gamma_t \end{aligned}$$

where

$$\gamma_s = \frac{\sigma^2 X_{s-}^2 E[\frac{\partial H}{\partial x}(x_T) D_{0,T} \mid F_{s-}] D_{0,s-}^{-1} + \sum_{i=1}^n \lambda_s^i a_s^i \Delta_s^i V}{\sigma^2 X_{s-}^2 + \sum_{i=1}^n \lambda_s^i (a_s^i)^2}$$

and

$$\Delta_s^i V = E[H(\xi_{s-,T}(X_{s-} + a_s^i)) - H(\xi_{s-,T}(X_{s-})) \mid X_{s-}].$$

From (18)

$$\begin{aligned} X_t = \xi_{s-,t}(X_{s-}) = \\ X_{s-} \exp\Big(\sigma(B_t - B_s) - \frac{\sigma^2}{2}(t-s) - \sum_{i=1}^{n} \int_s^t a_s^i \lambda_s^i ds\Big) \\ \times \prod_{s \le r \le t} (1 + a_r^i \Delta N_r^i). \end{aligned} \quad (19)$$

so

$$\begin{aligned} D_{s-,T} = D_{0,T} \cdot D_{0,s-}^{-1} = \\ \exp\Big(\sigma(B_t - B_s) - \frac{\sigma^2}{2}(t-s) - \sum_{i=1}^{n} \int_s^t a_s^i \lambda_s^i ds\Big) \\ \times \prod_{s \le r \le t} (1 + a_r^i \Delta N_r^i). \end{aligned}$$

Suppose $H(X_T)$ is a call option of the form $H(X_T) = (X_T - K)^+$. Then H is not C^2 but is the limit of the smooth functions $H_\varepsilon(X_T) = \frac{1}{2}(X_T - K + \sqrt{(X_T - K)^2 + \varepsilon})$; using approximation arguments it is shown, for example in [4], that the above theory applies to H. Now $\frac{\partial H}{\partial x}(X_T) = I_{X_T \ge K}$, so

$$E\Big[\frac{\partial H}{\partial x}(X_T) D_{0,T} \mid F_s\Big] D_{0,s-}^{-1} = E[I_{X_T \ge K} D_{s-,T} \mid X_{s-}]$$

and because the B and N^i are independent this can be evaluated as in [1], giving a Black–Scholes type formula. Similarly, with $\xi_{s-,T}(X_{s-})$ given by (19) $\Delta_s^i V$ can be calculated.

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